The Weingarten Calculus

Math 595

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Outline

Problem Statement

Given a compact group G, a finite dimensional Hilbert space $\mathcal{H}$, with orthonormal basis $\{e_1, \ldots, e_N\}$, and a continuous group homomorphism $U : \mathrm{G} \to \mathrm{U}(\mathcal{H})$, we wish to compute **Weingarten** integrals:

$$I_{ij} = \int_G \prod_{x=1}^{d} U_{i(x)j(x)}(g)dg$$

for some $i, j \in \mathrm{Fun}(d, N)$ where $U_{ab}(g) := \langle e_a, U(g)e_b \rangle$

Main Idea: Convert integration into computing entries of a projection onto the G-invariant subspace of $\mathcal{H}^{\otimes d}$

The Fundamental Theorem

Theorem

For any $d \in \mathbb{N}$ and $i, j \in Fun(d, N)$, we have

$$I_{ij} = \sum_{x,y}^{m} \mathbf{A}_{ix} \mathbf{W}_{xy} \mathbf{A}^{\star}_{yj}$$

Proof.

We prove by hand:

1. Linearize integrand
2. Interpret integral as orthogonal projection onto invariant subspace
3. Factor the projection given a basis of the invariant subspace
4. Read off entries of projection operator

□

The Fundamental Theorem, Proof continued

Defining $U_{ij}^{\otimes d}(g) := \langle e_i, U^{\otimes d}(g) e_j \rangle$, we see

$$I_{ij} = \int_G \prod_{x=1}^{d} U_{i(x)j(x)}(g) dg = \int_G U_{ij}^{\otimes d}(g) dg$$

Unraveling the definition of $U_{ij}^{\otimes d}(g)$,

$$\begin{aligned} U_{ij}^{\otimes d}(g) &= \langle e_{i(1)} \otimes \cdots \otimes e_{i(d)}, U(g) e_{j(1)} \otimes \cdots \otimes U(g) e_{j(d)} \rangle \\ &= \langle e_{i(1)}, U(g) e_{j(1)} \rangle \ldots \langle e_{i(d)}, U(g) e_{j(d)} \rangle \\ &= \prod_{x=1}^{d} \langle e_{i(1)}, U(g) e_{j(1)} \rangle \\ &= \prod_{x=1}^{d} U_{i(x)j(x)}(g) \end{aligned}$$

The Fundamental Theorem, Proof continued

Define $P = \int_G U^{\otimes d}(g) dg$. Then, $I_{ij} = P_{ij}$. The Haar measure forces P to be an orthogonal projection:

$$\begin{aligned} P^2 &= \int_G \int_G U^{\otimes d}(g) U^{\otimes d}(h) dg\, dh \\ &= \int_G \left(\int_G U^{\otimes d}(gh) \right) dg\, dh \\ &= \int_G \left(\int_G U^{\otimes d}(g) \right) dg\, dh \quad \text{by right invariance} \\ &= \int_G P\, dh \\ &= P \end{aligned}$$

The Fundamental Theorem, Proof continued

P projects $\mathcal{H}^{\otimes d}$ onto the space of G-invariant tensors:

$$(\mathcal{H}^{\otimes d})^G = \{v \in \mathcal{H}^{\otimes d} | U^{\otimes d}(g)v = v \quad \forall g \in G\}$$

- $\text{im}(P) \subseteq (\mathcal{H}^{\otimes d})^G$
  Let $h \in G, v \in \mathcal{H}^{\otimes d}$. Then,

$$\begin{aligned} U^{\otimes d}(h)Pv &= \left(\int_G U^{\otimes d}(h)U^{\otimes d}(g)dg\right) v \\ &= \left(\int_G U^{\otimes d}(hg)dg\right) v \\ &= \left(\int_G U^{\otimes d}(g)dg\right) v \\ &= Pv \end{aligned}$$

  so Pv is fixed by G for all $v \in \mathcal{H}^{\otimes d}$

The Fundamental Theorem, Proof continued

- $\mathrm{im}(P) \supseteq (\mathcal{H}^{\otimes d})^G$
  Let $v \in (\mathcal{H}^{\otimes d})^G$, so $U^{\otimes d}(g)v = v$ for all $g \in G$.

$$\begin{aligned} Pv &= \int_G U^{\otimes d}(g) v \, dg \\ &= \int_G v \, dg \\ &= v \end{aligned}$$

  so $v \in \mathrm{im}(P)$

The Fundamental Theorem, Proof continued

Let $\{a_1, \ldots, a_m\}$ be a basis for the G-invariant subspace of $\mathcal{H}^{\otimes d}$. Define

$$\mathbf{A} = [\langle e_i, a_x \rangle]_{i \in \mathrm{Fun}(d,N), x \in [m]}$$

Then, we can factor P:

$$P = \mathbf{A}(\mathbf{A}^\star \mathbf{A})^{-1} \mathbf{A}^\star$$

where the authors denote $\mathbf{W} = (\mathbf{A}^\star \mathbf{A})^{-1}$ as the **Weingarten** matrix. So, $P = \mathbf{A}\mathbf{W}\mathbf{A}^\star$. In particular,

$$P_{ij} = \sum_{x,y=1}^{m} \mathbf{A}_{ix} \mathbf{W}_{xy} \mathbf{A}^\star_{yj}$$

Toy Example - The Symmetric Group

Suppose we wish to calculate

$$I_{ij} = \int_{S(N)} \prod_{x=1}^{d} U_{i(x)j(x)}(g)dg$$
$$= \frac{1}{N!} \sum_{g \in S(N)} \prod_{x=1}^{d} U_{i(x)j(x)}(g)$$

Where we take $U : S(N) \to \mathrm{U}(\mathcal{H})$ to be the permutation representation:

$$U(\sigma)e_x = e_{\sigma(y)}$$

Toy Example - The Symmetric Group

The Fundamental Theorem states we need a basis for $(\mathcal{H}^{\otimes d})^{S(N)}$. **Claim:** for $\lambda \in \mathrm{Par}_N(d)$,

$$a_\lambda = \sum_{\substack{i \in \mathrm{Fun}(d,N) \\ \mathrm{type(i)}=\lambda}} e_i$$

form a basis for $(\mathcal{H}^{\otimes d})^{S(N)}$.
Here, type(i) is simply the (unordered) set of fibers (pre-images) of i ignoring empty fibers.
For example, $i = (1, 1, 2)$ and $j = (2, 2, 3)$ are equivalent types, but $i = (1, 2, 1)$ and $(1, 2, 3)$ are not.

Toy Example - The Symmetric Group

The a_λ are pairwise orthogonal:

$$\begin{aligned}\langle a_\lambda, a_\mu \rangle &= \left\langle \sum_{i \in \mathrm{Fun}(d,N)} \delta_{\mathrm{type}(i)\lambda} e_i, \sum_{j \in \mathrm{Fun}(d,N)} \delta_{\mathrm{type}(j)\lambda} e_j \right\rangle \\ &= \sum_{i,j \in \mathrm{Fun}(d,N} \delta_{\mathrm{type}(i)\lambda} \delta_{\mathrm{type}(j)\mu} \delta_{ij} \\ &= \delta_{\lambda\mu} \sum_{i \in \mathrm{Fun}(d,N)} \delta_{\mathrm{type}(i)\lambda} \\ &= \delta_{\lambda\mu} \binom{N}{\#(\lambda)}\end{aligned}$$

where $\#(\lambda)$ is the number of blocks of the partition λ.

Toy Example - The Symmetric Group

So, $W_{\lambda\mu} = (\mathbf{A}^\star \mathbf{A})^{-1}_{\lambda\mu} = \delta_{\lambda\mu}\langle a_\lambda, a_\mu\rangle^{-1} = \delta_{\lambda\mu}\binom{N}{\#(\lambda)}^{-1}$

Finally, let $C_N(\lambda) = \binom{N}{\#(\lambda)}^{-1}$, then

$$\begin{aligned} I_{ij} &= \int_{S(N)} \prod_{x=1}^{d} U_{i(x)j(x)}(g)dg \\ &= \sum_{\lambda,\mu \in \mathrm{Par}_N(d)} \langle e_i, a_\lambda\rangle \mathbf{W}_{\lambda\mu}\langle a_\mu, e_j\rangle \\ &= \sum_{\lambda,\mu \in \mathrm{Par}_N(d)} \delta_{\mathrm{type(i)},\lambda}\delta_{\lambda\mu}\delta_{\mu,\mathrm{type(j)}} C_N(\lambda) \\ &= \delta_{\mathrm{type(i)},\mathrm{type(j)}} C_N(\mathrm{type(i)}) \end{aligned}$$

The Unitary Group

Under the tautological representation, all Weingarten integrals vanish. Consider instead the **adjoint** representation:

- $\mathrm{End}(\mathcal{H})$ is the representation space.
- $\langle A, B\rangle = \mathrm{Tr} A^\star B$ is the inner product
- The representation is $V : \mathrm{U}(N) \to \mathrm{U}(\mathrm{End}(\mathcal{H}))$ with action defined by

$$V(g)A = g^{-1}Ag$$

The adjoint representation is in fact related to the tautological representation via the identity

$$V_{yy'xx'} = \overline{U_{xy}(g)}U_{x'y'}(g)$$

Thus, we recover Weingarten integrals of the form

$$I_{ii'jj'} = \int_{\mathrm{U}(N)} \prod_{x=1}^{d} \overline{U_{i(x)i'(x)}(g)}U_{j(x)j'(x)}(g)dg$$

The Unitary Group, Enter Schur-Weyl Duality

Recall that the Fundamental Theorem requires we find a basis for the U(N)-invariant subspace:

$$\begin{aligned}(\mathrm{End}(\mathcal{H}^{\otimes d}))^{\mathrm{U}(N)} &= \{T \in \mathrm{End}(\mathcal{H}^{\otimes d}) | V(g)^{\otimes d} T = T, \quad \forall g \in U(N)\} \\ &= \{T \in \mathrm{End}(\mathcal{H}^{\otimes d}) | T g^{\otimes d} = g^{\otimes d} T, \quad \forall g \in U(N)\}\end{aligned}$$

But Schur-Weyl duality says that (Proposition 4.10 in lecture notes)

$$(\mathrm{End}(\mathcal{H}^{\otimes d}))^{\mathrm{U}(N)} = \mathrm{span}(\{A_\sigma | \sigma \in S_d\})$$

where

$$A_\sigma v_1 \otimes \cdots \otimes v_d = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \quad \sigma \in S_d$$

The Unitary Group, Enter Schur-Weyl Duality

Observing that

$$\begin{aligned}\langle A_\rho, A_\sigma \rangle &= \mathrm{Tr} A_\rho^\star A_\sigma \\ &= \mathrm{Tr} A_{\rho^{-1}} A_\sigma \\ &= \mathrm{Tr} A_{\rho^{-1}\sigma} \\ &= N^{\#\text{cycles}(\rho^{-1}\sigma)}\end{aligned}$$

So, we see that

$$\mathbf{A}^\star \mathbf{A} = [N^{\#\text{cycles}(\rho^{-1}\sigma)}]_{\rho,\sigma\in S_d}$$

as desired.

The Unitary Group

Important

If $d > N$, then the A_σ are linearly dependent, so $\mathbf{A}^\star\mathbf{A}$ is singular. However, Baik and Rains showed that

$$\{A_\sigma | \sigma \in S_N(d)\}$$

is always a basis.

Finally, the goal is to determine the Weingarten Matrix $\mathbf{W} = (\mathbf{A}^\star\mathbf{A})^{-1}$ so that we have

$$I_{ii'jj'} = \sum_{\rho,\sigma \in S_d} \mathbf{A}_{ii',\rho}\mathbf{W}_{\rho\sigma}\mathbf{A}_{\sigma,jj'}$$

.

The Unitary Group

Theorem

For any $\rho, \sigma \in S_d$, we have that

$$\mathbf{W}_{\rho\sigma} = \sum_{\lambda \vdash d} \frac{\chi^\lambda(\rho^{-1}\sigma)}{\prod_{\square \in \lambda}(N + c(\square))} \frac{dim V^\lambda}{d!}$$

where χ^λ is the character of V^λ.

Finally, it can be shown that

$$\boxed{I_{ii',jj'} = \sum_{\rho,\sigma \in S_d} \delta_{i,i' \circ \rho} \mathbf{W}_{\rho\sigma} \delta_{j,j' \circ \sigma}}$$

Example Calculation

For $N \geq 3$, show that

$$I_{ii'jj'} = \int_{\mathrm{U}(N)} \overline{U_{11} U_{22} U_{33}} U_{12} U_{23} U_{31} \, \mathrm{dg} = \frac{2}{N(N^2-1)(N^2-4)}.$$

Reference Definitions and Facts

For $I_{ii'jj'} = \int_{\mathrm{U}(N)} \overline{U_{11}\,U_{22}\,U_{33}}\,U_{12}\,U_{23}\,U_{31}\;\mathrm{dg}$, we have

$$i = \mathrm{id}, \quad i' = \mathrm{id}, \quad j = \mathrm{id}, \quad j' = (321).$$

$$I_{ii'jj'} = \sum_{\rho\in S_n}\sum_{\sigma\in S_n} \delta_{i,i'\circ\rho}\,\delta_{j,j'\circ\sigma}\,W_{\rho\sigma}.$$

$$W_{\rho\sigma} = \sum_{\lambda\vdash n} \frac{\chi_\lambda(\rho^{-1}\sigma)}{\prod_{\Box\in\lambda}(N + c(\Box))} \cdot \frac{\dim V_\lambda}{d!}.$$

$$\dim V_\lambda = \frac{d!}{\prod_{(i,j)\in\lambda} h(i,j)}.$$

Character Table of S_3

Representation/Conjugacy class representative	() (identity element) -- size 1	$(1, 2, 3)$ (3-cycle) -- size 2	$(1, 2)$ (2-transposition) -- size 3
Trivial representation	1	1	1
Unknown other representation of degree one	1	1	-1
Unknown other represehtation of degree two	2	-1	0

Figure: Character Table for S_3. Source (groupprops.subwiki.org)

Weingarten to Free Probability

- Reinterpret this in the language of random variables for non-commutative matrices
- What happens to $Wg(N, \sigma)$ for big N
- See that in the large N limit, Weingarten sums turn into free probability counterparts.

Noncommutative Probability Spaces

A **noncommutative probability space** is a pair $(\mathcal{A}, \varphi)$:

- $\mathcal{A}$: unital $*$-algebra (where our randomness lives);
- $\varphi : \mathcal{A} \to \mathbb{C}$: state (allows us the role of expectation).

Recall:

$$\mathcal{A} = M_N(\mathbb{C}), \qquad \varphi(a) = \frac{1}{N}\mathrm{Tr}(a).$$

- $\varphi(I) = 1$ (normalization).
- $\varphi(\alpha a + \beta b) = \alpha\varphi(a) + \beta\varphi(b)$ (linearity).
- If a has eigenvalues $\lambda_1, \ldots, \lambda_N$, then

$$\varphi(a^k) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^k \quad = k\text{-th moment of empirical eigenvalue law.}$$

Matrices are now our "random variables"; the normalized trace is our "expectation". For our noncommutativity ($ab \neq ba$) setting.

Free Independence

Definition

Unital sub-algebras $(\mathcal{A}_j)_{j \in J}$ of $(\mathcal{A}, \varphi)$ are **freely independent** if for all $m \geq 1$ and all $a_\ell \in \mathcal{A}_{j_\ell}$ with

$$\varphi(a_\ell) = 0, \qquad j_1 \neq j_2 \neq \cdots \neq j_m,$$

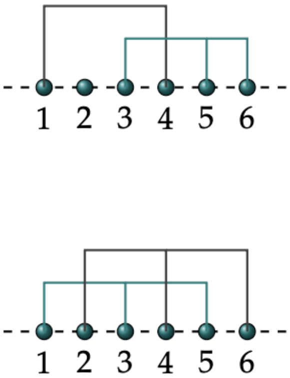
we have

$$\varphi(a_1 a_2 \cdots a_m) = 0.$$

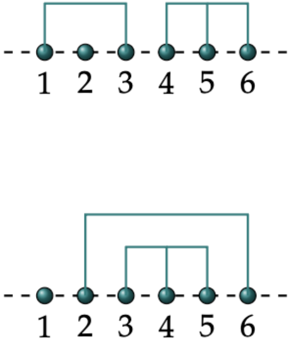
Note: freeness is the analog of independence for non-commutative random variables. Think of it as coming from the free product of algebras.

"Free probability $\approx$ noncommutative probability $+$ free independence"

Noncrossing Partitions



Crossing Partitions



Non-Crossing Partitions

Semicircles, Noncrossing Partitions & Catalan

Free CLT / semicircle law.

- In free probability, the analogue of the Gaussian is the semicircular law μ_{sc} on $[-2, 2]$.
- Its even moments are Catalan numbers:

$$\int x^{2k}\, d\mu_{\mathrm{sc}}(x) = C_k = \frac{1}{k+1}\binom{2k}{k}, \quad \int x^{2k+1}\, d\mu_{\mathrm{sc}}(x) = 0.$$

Noncrossing partitions.

- A partition with no crossing arcs is *noncrossing*; set of these is $\mathrm{NC}(n)$.
- Observe: $\#\mathrm{NC}(n) = C_n$.

Observe: Catalan numbers appear both in semicircle moments and in counts of noncrossing partitions; we will see them again inside the Weingarten function.

Asymptotics for the Weingarten

Theorem (Eq. 3.7 Collins 2015)

for large N:

$$\mathrm{Wg}(N,\sigma) = \underbrace{N^{-(p+|\sigma|)}}_{\textit{power of } N} \times \Big(\underbrace{\mathrm{Mob}(\sigma)}_{\substack{\textit{leading}\\ \textit{coefficient}}} + \underbrace{O(N^{-2})}_{\substack{\textit{smaller}\\ \textit{corrections}}}\Big).$$

where

- $|\sigma|$ is the Cayley Length
- As $N \to \infty$, only the permutations with smallest possible $|\sigma|$ survive. These are our geodesic permutations.

Weingarten Asymptotics and Noncrossing Partitions

Recall the asymptotic for fixed $\sigma \in S_p$:

$$\mathrm{Wg}(N, \sigma) = N^{-(p+|\sigma|)}\Big(\mathrm{Mob}(\sigma) + O(N^{-2})\Big).$$

(Collins 2015):

- $\mathrm{Mob}(\sigma) = 0$ unless σ is *geodesic*:

$$|\sigma| = p - \#\text{cycles}(\sigma^{-1}\gamma),$$

  where $\gamma = (1\ 2\ \ldots\ p)$ is the full cycle.
- Geodesic permutations are in bijection with *noncrossing partitions* of $\{1, \ldots, p\}$.
- On this subset, $\mathrm{Mob}(\sigma)$ agrees with the Mobius function on the lattice $\mathrm{NC}(p)$, and can be written as a product of Catalan numbers.

So this gives us that:

$$\text{noncrossing partitions} \quad \leftrightarrow \quad \text{free cumulants.}$$

Asymptotic Freeness

Theorem (General Case: DVV 1990s/ Collins 2015 Thm 3.10)

Let $U_1^{(d)}, \ldots, U_r^{(d)}$ be independent Haar unitaries in $M_d(\mathbb{C})$, and let $W_1^{(d)}, \ldots, W_s^{(d)}$ be deterministic matrices with limiting eigenvalue distributions. Then as $d \to \infty$:

$$U_1^{(d)}, \ldots, U_r^{(d)}, W_1^{(d)}, \ldots, W_s^{(d)} \xrightarrow{*\text{-dist}} \textit{freely independent variables}$$

Questions?

- The Weingarten Calculus (Collins 2021)