Trace norm and trace distance

Definition (Trace norm)

The *trace norm* of a linear operator $X \in \mathcal{L}(\mathcal{H})$ is

$$\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X} = \sum_{i=1}^{d} s_i(X),$$

where $d = \dim\mathcal{H}$ and $s_i(X)$ are the singular values of X.

Trace distance

The *trace distance* of quantum states ρ and σ is defined as

$$D(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1 .$$

Fidelity

The fidelity $F(\rho, \sigma)$ of quantum states ρ and σ is defined as

$$F(\rho, \sigma) = \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1 = \mathrm{tr}\left(\sigma^{1/2}\rho\sigma^{1/2}\right)^{1/2} .$$

Properties of trace distance and fidelity

(i) $D(\cdot,\cdot)$ is a **metric** (non-negative, symmetric and satisfying triangle inequality), but fidelity is not. We do have $F(\rho,\sigma) = F(\sigma,\rho) \geq 0$.

(ii) $0 \leq D(\rho,\sigma) \leq 1$ and $D(\rho,\sigma) = 0$ iff $\rho = \sigma$, while $D(\rho,\sigma) = 1$ iff $\mathrm{supp}(\rho) \perp \mathrm{supp}(\sigma)$.
$0 \leq F(\rho,\sigma) \leq 1$ and $F(\rho,\sigma) = 1$ iff $\rho = \sigma$, while $F(\rho,\sigma) = 0$ iff $\mathrm{supp}(\rho) \perp \mathrm{supp}(\sigma)$.

(iii) $D(\rho,\sigma) = \sup\{\mathrm{tr}[P(\rho-\sigma)] : P \geq 0 \text{ and } \mathbb{1} - P \geq 0\}$.

(iv) **Data-processing:**

$$D(\rho,\sigma) = D(U\rho U^\dagger, U\sigma U^\dagger) \text{ and } F(\rho,\sigma) = F(U\rho U^\dagger, U\sigma U^\dagger) \text{ for all unitaries } U;$$

$$D(\rho_A,\sigma_A) \leq D(\rho_{AB},\sigma_{AB}) \text{ and } F(\rho_{AB},\sigma_{AB}) \leq F(\rho_A,\sigma_A).$$

(v) Let ρ_i, σ_i be quantum states and $(p_i)_i$ a probability distribution.
$D(\cdot,\cdot)$ is **jointly convex**,

$$D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \leq \sum_i p_i D(\rho_i,\sigma_i),$$

while $F(\cdot,\cdot)$ is **jointly concave**,

$$F\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \geq \sum_i p_i F(\rho_i,\sigma_i).$$

Properties of trace distance and fidelity

(i) **Holevo-Helstrom theorem:** $D(\rho, \sigma)$ is related to the maximum success probability $p_{\text{succ}}(\rho, \sigma)$ of distinguishing between ρ and σ,

$$p_{\text{succ}}(\rho, \sigma) = \frac{1}{2}\left(1 + D(\rho, \sigma)\right).$$

(ii) **Uhlmann's theorem:**

$$F(\rho, \sigma) = \max\{|\langle \psi^{\rho} | \phi^{\sigma} \rangle|$$

where the maximization is over purifications $|\psi^{\rho}\rangle, |\phi^{\sigma}\rangle$ of ρ, σ, respectively.

(iii) **Fuchs-van-de-Graaf inequalities:**

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}.$$