Recap: Compact groups and Haar measure

Definition

A *topological group* is a group G endowed with a topology such that group multiplication and inversion are continuous. A *compact group* is a topological group that is compact, that is, every open cover of G has a finite subcover. Closed subgroups of a compact group are also compact groups.

Haar measure

Let G be a compact group. There exists a unique measure $\mathrm{d}g$ on G, called the *Haar measure*, satisfying the following properties:

1. Invariance: for every continuous function $f : G \to \mathbb{C}$ and every $h \in G$,

$$\int_G f(g)\mathrm{d}g = \int_G f(gh)\mathrm{d}g = \int_G f(hg)\mathrm{d}g\,.$$

2. Normalization: $\int_G 1\mathrm{d}g = 1$.

Generalization of group average: $\frac{1}{|G|}\sum_{g\in G} f(g)$ for finite $G \longrightarrow \int_G \mathrm{d}g\, f(g)$ for compact G.

Recap: Representation theory of compact groups

Using the Haar measure, one can prove analogous statements about finite-dimensional representations of compact groups, e.g.:

1. Every G-invariant subspace has a G-invariant complement.
2. Every representation over $\mathbb{C}$ decomposes as a sum of irreducible representations.
3. Most aspects of character theory also carry over to the compact case.

The regular representation of a compact group G is defined as the Hilbert space $L^2(G)$ of square integrable functions on G, with the action of G given by $\varphi(g)(f)\colon\ h \mapsto f(g^{-1}h)$.

Peter-Weyl theorem

(i) The linear span of all matrix coefficients of the irreducible unitary representations of G is dense in $L^2(G)$.

(ii) Every irreducible unitary representation of G is finite-dimensional.

(iii) The regular representation (which has infinite dimension if G is not finite) $L^2(G)$ decomposes into a direct sum of the irreducible unitary representations of G, each occurring with multiplicity equal to its dimension. The matrix coefficients of the complete set of irreps form an orthonormal basis of $L^2(G)$.

Recap: Commutants

External product representation

Let (φ, V) and (ψ, W) be representations of groups G and H respectively. Then $V \hat{\otimes} W$ affords the *external product representation* of the direct product $G \times H$ by defining

$$(\varphi \hat{\otimes} \psi)(g, h) := \varphi(g) \otimes \psi(h).$$

Commutant

Let $\mathcal{S}$ be a subset of an algebra $\mathcal{A}$. The commutant $\mathcal{S}'$ of $\mathcal{S}$ is the collection of those elements in $\mathcal{A}$ commuting with all of $\mathcal{S}$:

$$\mathcal{S}' = \{a \in \mathcal{A} : as = sa \text{ for all } s \in \mathcal{S}\}.$$

Lemma

Let V and W be finite-dimensional complex vector spaces. The commutant of $\mathrm{End}(V) \otimes \mathbb{1}_W$ in $\mathrm{End}(V \otimes W) \cong \mathrm{End}(V) \otimes \mathrm{End}(W)$ is $\mathbb{1}_W \otimes \mathrm{End}(W)$.

Annonucement

Thursday, Oct 9

Interactive problem session led by Jacob and Theshani (instead of class).

Exercises to be discussed: 3.1, 3.5, 3.6, 3.9, and 3.12.

Typed-up solutions will be provided afterwards.