Constructing new representations

Tensor representation

Let (φ, V) and (ψ, W) be representations of a group G. Then $(\varphi \otimes \psi)(g) \coloneqq \varphi(g) \otimes \psi(g)$ defines a representation on $V \otimes W$ called the *tensor representation*.

$V \otimes W$ is in general *reducible*, even for irreducible V, W.

Dual representation

Let (φ, V) be a representation of G. Let V^* be the dual space of V consisting of the vector space of linear maps from V to $\mathbb{C}$. The *dual* representation (φ^*, V^*) is defined as

$$\varphi^*(g)(L) \coloneqq L \circ \varphi(g)^{-1} \quad \text{for } g \in G \text{ and } L \in V^*.$$

Hom-space representation

Let (φ, V) and (ψ, W) be two representations of a group G. Then G acts on $\mathrm{Hom}(V, W)$ by sending $f : V \to W$ to $\psi(g) \circ f \circ \varphi(g)^{-1}$, which turns $\mathrm{Hom}(V, W)$ into a representation of G.

Group algebra

Group algebra $\mathbb{C}[G]$ is the vector space V generated by $\{|g\rangle\}_{g\in G}$ together with group action $\varphi_R(g)|h\rangle = |gh\rangle$ and multiplication

$$\left[\sum_{g\in G} c_g |g\rangle\right] \cdot \left[\sum_{h\in G} d_h |h\rangle\right] = \sum_{g,h\in G} c_g d_h |gh\rangle = \sum_{g\in G} f_g |g\rangle \quad \text{with} \quad f_g = \sum_{h\in G} c_{gh^{-1}} d_h.$$

Elements $\sum_{g\in G} c_g |g\rangle$ of $\mathbb{C}[G] \longleftrightarrow$ functions $f : G \to \mathbb{C}$ via $g \mapsto c_g$

A function $f : G \to \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes of G:

$$f(g) = f(hgh^{-1}) \quad \text{for all } g, h \in G.$$

Character of a representation

Let (φ, V) be a representation of G. The *character* $\chi = \chi_V$ of (φ, V) is the class function defined by $\chi(g) = \mathrm{tr}(\varphi(g))$.

Properties of characters

Let (φ, V) and (ψ, W) be representations of a group G with identity element e, and denote by χ_V and χ_W the associated characters.

(i) $\chi_V(e) = \mathrm{tr}(\mathbb{1}_V) = \dim V$ is the degree of the representation (φ, V).

(ii) If (φ, V) is unitary, then $\chi(g^{-1}) = \overline{\chi(g)}$.

(iii) $\chi_{V \oplus W} = \chi_V + \chi_W$.

(iv) $\chi_{V \otimes W} = \chi_V \chi_W$.

(v) The characters of irreducible representations form an ONB for the set of class functions.

(vi) If W is irreducible, then the projection onto the isotypical component corresponding to W is given by

$$P_W = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \varphi(g).$$