Recap: Schmidt decomposition

Let $|\psi\rangle_{AB}$ be a pure bipartite quantum state. Then there are sets of orthonormal vectors $\{|e_i\rangle_A\}_{i=1}^r$ and $\{|f_j\rangle_B\}_{j=1}^r$ and strictly positive real numbers $(\lambda_i)_{i=1}^r$ such that

$$|\psi\rangle_{AB} = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |e_i\rangle_A \otimes |f_i\rangle_B.$$

The *Schmidt coefficients* $(\lambda_i)_{i=1}^r$ satisfy $\sum_{i=1}^r \lambda_i = 1$, and are unique up to reordering. The integer r is called the *Schmidt rank* of $|\psi\rangle_{AB}$.

The state $|\psi\rangle_{AB}$ is entangled iff $r > 1$. The marginals of $|\psi\rangle_{AB}$ are given by

$$\rho_A = \mathrm{tr}_B \psi_{AB} = \sum_{i=1}^{r} \lambda_i |e_i\rangle\langle e_i|_A \qquad \rho_B = \mathrm{tr}_A \psi_{AB} = \sum_{i=1}^{r} \lambda_i |f_i\rangle\langle f_i|_B.$$

These are spectral decompositions, that is, ρ_A and ρ_B have the same spectrum given by the Schmidt coefficients, and the *Schmidt vectors* $\{|e_i\rangle_A\}$ and $\{|f_j\rangle_B\}$ can be completed to eigenbases of ρ_A and ρ_B, respectively.

Recap: Purifications

Purification

Let ρ_A be a mixed quantum state. Any state $|\psi\rangle_{AR} \in \mathcal{H}_A \otimes \mathcal{H}_R$ satisfying

$$\operatorname{tr}_R \psi_{AR} = \rho_A,$$

where $\mathcal{H}_R$ is some auxiliary Hilbert space, is called a *purification* of ρ_A.

Purifications exist and are equivalent

Let ρ_A be a mixed quantum state.

(i) There exists a purification of ρ_A on $\mathcal{H}_A \otimes \mathcal{H}_R$ with $\dim \mathcal{H}_R \geq \operatorname{rank} \rho_A$.

(ii) Any two purifications are isometrically equivalent: Let $|\psi\rangle_{AR_1}$ and $|\varphi\rangle_{AR_2}$ be two purifications of ρ_A, and without loss of generality assume $\dim \mathcal{H}_{R_1} \leq \dim \mathcal{H}_{R_2}$. Then there exists an isometry $V : \mathcal{H}_{R_1} \to \mathcal{H}_{R_2}$ such that

$$|\varphi\rangle_{AR_2} = \left(\mathbb{1}_A \otimes V\right) |\psi\rangle_{AR_1}.$$