

Recap: Decomposing representations

Invariant complements

Let (φ, V) be a representation of a finite group G for which V is a vector space over a field whose characteristic does not divide the order of G . Then every G -invariant subspace W has a G -invariant complement W' , i.e., $V = W \oplus W'$ (as vector spaces and as representations).

Complete reducibility

A representation is called *completely reducible* if it is equivalent to a direct sum of irreducible representations.

Maschke's theorem

Every finite-dimensional representation of a finite group G over a field with characteristic not dividing $|G|$ is completely reducible.

Schur's Lemma

Schur's lemma

Let (φ_1, V_1) and (φ_2, V_2) be irreducible representations of a group G , and let $f: V_1 \rightarrow V_2$ be a G -equivariant linear map, that is, $f \cdot \varphi_1(g) = \varphi_2(g) \cdot f$ for all $g \in G$. Then the following hold:

- (i) Either f is invertible (and hence $V_1 \cong V_2$) or $f = 0$.
- (ii) If $V_1 = V_2$ is finite-dimensional over an algebraically closed field \mathbb{F} (for example $\mathbb{F} = \mathbb{C}$), then $f = \lambda \mathbb{1}_{V_1}$ for some $\lambda \in \mathbb{F}$.

Corollary

Let (φ, V) be an irreducible representation of a group G . Then any operator X satisfying $\varphi(g)X = X\varphi(g)$ for all $g \in G$ is proportional to the identity.

Irreps of abelian groups are 1-dimensional

Let G be an abelian group. Then any complex irreducible representation of G is one-dimensional.

Isotypical decomposition

Definition (Isotypical decomposition)

Let (φ, V) be a finite-dimensional representation of a finite group G over \mathbb{C} . Consider a decomposition $V \cong \bigoplus_k V_k$ and $\varphi \cong \bigoplus_k \varphi_k$ with the following properties:

- Each (φ_k, V_k) is the direct sum of n_k copies of an irrep (ψ_k, W_k) of G :

$$V_k = W_k \oplus \dots \oplus W_k = W_k^{\oplus n_k} \cong W_k \otimes \mathbb{C}^{n_k}$$

$$\varphi_k = \psi_k \oplus \dots \psi_k = \psi_k \otimes \text{id}_{\mathbb{C}^{n_k}}.$$

- The different W_k are inequivalent, that is, $W_k \not\cong W_{k'}$ for $k \neq k'$.

Then V_k is called an *isotypical component*, and $V = \bigoplus_k V_k$ (with $\varphi = \bigoplus_k \varphi_k$) is called *isotypical decomposition*.

Uniqueness of isotypical components

The decomposition $V = \bigoplus_k V_k$ of a representation V into isotypical components V_k is unique, and so are the multiplicities n_k of W_k in V_k .