Recap: Invariant subspaces

Invariant subspaces and subrepresentations

Let (φ, V) be a representation of a group G. A subspace $W \subset V$ is called *G-invariant* if

$$\varphi(g)|w\rangle \in W \quad \text{for all } |w\rangle \in W \text{ and } g \in G.$$

The restriction $\varphi|_W$ of φ onto W is called a *subrepresentation*.

Given: representation (ψ, V) of a finite group G of degree $m = \dim V$, and a G-invariant subspace $W \leq V$ of dimension $k = \dim W$.

Choose a basis $\{w_1, \ldots, w_k, w_{k+1}, \ldots, w_m\}$ for V such that $W = \operatorname{span}(w_1, \ldots, w_k)$, and set $W' = \operatorname{span}(w_{k+1}, \ldots, w_m)$ so that $V = W \oplus W'$.

Then every $\psi(g)$ has the following representation matrix with respect to this basis:

$$\psi(g) = \begin{array}{c} \begin{matrix} \quad W & \quad\quad W' \end{matrix} \\ \left(\begin{array}{c|c} \psi(g)|_W & * \\ \hline 0 & * \end{array}\right) \begin{matrix} W \\ W' \end{matrix} \end{array}$$

Recap: Irreducible representations

Irreducible representation

A representation (φ, V) of a group G is called *irreducible* if $\{0\}$ and V are the only G-invariant subspaces of V.

Direct sums of representations

Let (φ_1, V_1) and (φ_2, V_2) be representations of a group G. Then the vector space direct sum $V_1 \oplus V_2$ affords the of G-representation

$$[(\varphi_1 \oplus \varphi_2)(g)](v_1 \oplus v_2) \coloneqq [\varphi_1(g)](v_1) \oplus [\varphi_2(g)](v_2).$$

This is called the *direct sum* of the representations (φ_1, V_1) and (φ_2, V_2).

Invariant complements

Let (φ, V) be a representation of a finite group G for which V is a vector space over a field whose characteristic does not divide the order of G. Then every G-invariant subspace W has a G-invariant complement W', i.e., $V = W \oplus W'$ (as vector spaces and as representations).