Recap: Groups

Group

A group $(G, \cdot)$ is a set G together with a binary operation $\cdot : G \times G \to G$ satisfying:

- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.
- Identity element: there exists an element $e \in G$ with $e \cdot g = g \cdot e = g$ for all $g \in G$.
- Inverse: for all $g \in G$, there exists $h \in G$ with $g \cdot h = h \cdot g = e$. Such an element h is unique; it is called the *inverse* of g,and denoted by g^{-1}.

Examples

(i) $(V, +)$, where V is a vector space.

(ii) $(\mathbb{K}, +)$ and $(\mathbb{K}\backslash\{0\}, \cdot)$, where $(\mathbb{K}, +, \cdot)$ is a field.

(iii) **Symmetric group** $S_n = \{\text{bijections on } \{1, 2, ..., n\}\}$.

(iv) **General linear group** $\mathrm{GL}(V) = \{f \in \mathcal{L}(V) \text{ invertible}\}$.

(v) **Unitary group** $\mathcal{U}(V) = \{U \in \mathrm{GL}(V) : U^\dagger U = \mathbb{1}\}$.

Recap: Representations

Group homomorphism

Let $(G, \cdot)$ and $(H, *)$ be two groups. A group homomorphism $\varphi : G \to H$ is a function satisfying $\varphi(x \cdot y) = \varphi(x) * \varphi(y)$ for all $x, y \in G$.

Representation of a group

A representation (φ, V) of a group G on a vector space V over a field $\mathbb{F}$ is a group homomorphism $\varphi : G \to \mathrm{GL}(V)$.

Examples of representations

(i) **Trivial representation** $\varphi(g) = \mathbb{1}_V$ for all $g \in G$.

(ii) Let $G := \langle g : g^d = e \rangle$ be a **cyclic group** of order d generated by g. Let $V = \mathbb{C}^d$ with basis $|0\rangle, \dots, |d-1\rangle$ and define $g \mapsto X$, where $X|i\rangle = |i + 1 \bmod d\rangle$ for all i.

(iii) Alternatively, $g \mapsto Z$, where $Z|j\rangle = \omega^j |j\rangle$ for a primitive d-th root of unity ω.