

Recap: Dimension bounds

Schur-Weyl decomposition:

$$(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} V_\lambda \otimes U_\lambda^d.$$

Number of Young diagrams of at most d rows

$$|\{\lambda \vdash_d n\}| \leq (n+1)^d$$

Dimension bound for S_n -irreps

$$\exp(nH(\bar{\lambda})) (n+d)^{-d(d+1)/2} \leq \dim V_\lambda \leq \exp(nH(\bar{\lambda})),$$

with **Shannon entropy** $H(p) = -\sum_i p_i \log p_i$.

Dimension bound for \mathcal{U}_d -irreps

$$\dim U_\lambda^d \leq (n+1)^{d(d-1)/2}$$

Recap: Quantum method of types

Denote by Π_λ the isotypical projection onto the summand $V_\lambda \otimes U_\lambda^d$ in the Schur-Weyl decomposition.

Let $\rho \in \mathcal{L}(\mathbb{C}^d)$ be a quantum state of a d -dimensional system, and denote by $s = (s_1, \dots, s_d)$ the ordered spectrum of ρ .

Concentration of measure

$$\begin{aligned} \exp(-nD(\bar{\lambda} \| s)) (n+d)^{-d(d+1)/2} &\leq \text{tr}(\Pi_\lambda \rho^{\otimes n}) = \dim V_\lambda \text{tr}[\omega_\lambda(\rho)] \\ &\leq \exp(-nD(\bar{\lambda} \| s)) (n+d)^{d(d-1)/2}, \end{aligned}$$

where $D(p \| q) = \sum_i p_i \log(p_i / q_i)$ is the **relative entropy** between p and q .

Recap: Werner states

Definition

Let $\mathcal{H}_A = \mathcal{H}_B \cong \mathbb{C}^d$ be d -dimensional Hilbert spaces $d \geq 2$. A quantum state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called a *Werner state* [Wer89] if

$$(U \otimes U)\rho_{AB}(U \otimes U)^\dagger = \rho_{AB} \quad \text{for all } U \in \mathcal{U}_d.$$

Werner states

A Werner state has the form

$$\rho_{AB} = x \frac{2}{d(d+1)} P_{\square\square} + (1-x) \frac{2}{d(d-1)} P_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}} \quad \text{for some } x \in [0, 1],$$

where $P_{\square\square} = \frac{1}{2}(\mathbb{1} + \mathbb{F})$ and $P_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}} = \frac{1}{2}(\mathbb{1} - \mathbb{F})$.

$P_{\square\square}$ and $P_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}$ are the projections onto the two summands in $(\mathbb{C}^d)^{\otimes 2} = U_{\square\square}^d \oplus U_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}^d$.

Recap: Entanglement in Werner states

Visibility

A Werner state can be written as

$$\rho_{AB} = \frac{1}{d(d^2 - 1)} [(d - \alpha)\mathbb{1} + (d\alpha - 1)\mathbb{F}],$$

with the *visibility* $\alpha := \text{tr}(\rho_{AB}\mathbb{F})$.

Properties of Werner states

- (i) Every Werner state is invariant under twirling $\mathcal{T}(X) = \int_{\mathcal{U}_d} dU (U \otimes U)X(U \otimes U)^\dagger$.
- (ii) Let ρ_{AB} be an arbitrary state. Then $\mathcal{T}(\rho_{AB})$ is a Werner state of visibility $\alpha = \text{tr}(\mathbb{F}\rho_{AB})$.
- (iii) A Werner state ρ_{AB} is entangled iff $\alpha = \text{tr}(\rho_{AB}\mathbb{F}) < 0$.