Announcements

Office hours this week

Friday, Oct 17, 2pm in 3217 Everitt

Visitor: Daniel Spiegel (Harvard)

Visiting all week.

Seminar talk on Wed, Oct 22, 4-5pm in 1022 Lincoln Hall:

“A Classifying Space for Phases of Matrix Product States”

Recap: Conjugacy classes of S_n

Facts about permutations

(i) Every permutation $\pi \in S_n$ can be written uniquely as a product of disjoint cycles, e.g., $\pi = (13)(2)(465) \in S_6$. The *cycle type* of a permutation $\pi \in S_n$ is the tuple of cycle lengths in non-increasing order. For example, $\pi = (14)(236)(58)(7)$ has cycle type $(3, 2, 2, 1)$.

(ii) Cycle types $(\lambda_1, ..., \lambda_d)$ of a permutation $\pi \in S_n$ form an ordered partition of n, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$ and $\sum_{i=1}^{d} \lambda_i = n$. We use the notation $\lambda \vdash_d n$ for an ordered partition of n into at most d parts. Note: If $d < n$ then not all possible partitions or cycle types appear.

(iii) Two permutations $\pi, \pi' \in S_n$ are conjugate iff they have the same cycle type.

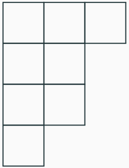
It follows from (i)-(iii) above that the conjugacy classes of S_n, and hence its irreducible representations, are indexed by the ordered partitions of n into n parts.

Recap: Partitions as diagrams

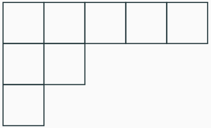
Definition

Let $\lambda = (\lambda_1, \ldots, \lambda_d) \vdash_d n$ be a partition of n into at most d parts. The *Young diagram* corresponding to $\lambda \vdash_d n$ is an arrangement of n boxes into d rows such that the i-th row has length λ_i.

Examples: two Young diagrams $\lambda_1, \lambda_2 \vdash_4 8$



$\lambda_1 = (3, 2, 2, 1) \vdash_4 8$



$\lambda_2 = (5, 2, 1, 0) \vdash_4 8.$

Recap: Young tableaux

Definition (Young tableaux)

- A *Young tableau* is a Young diagram λ with boxes labeled with numbers $\{1, \ldots, N\}$ (we can have $N \neq n$). We call λ the *shape* of the Young tableau T.
- A *standard Young tableau* is a Young tableau with $N = n$, and the labels are strictly increasing along rows (left to right) and along columns (top to bottom).
- A *semistandard Young tableau* is a Young tableau whose labels are non-decreasing along rows and strictly increasing along columns.

The standard Young tableaux of shape $\lambda = (3, 2)$ are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}.$$

The semistandard Young tableaux of shape $\lambda = (3, 2)$ with numbering $\{1, 2\}$ are

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}.$$

Recap: Combinatorial formulas for irrep dimensions

Let $d, n \in \mathbb{N}$. The number of standard Young tableaux of shape $\lambda \vdash_d n$ is equal to

$$d_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)},$$

where for a box (i,j) in row i and column j of λ we define the **hook length**

$$h(i,j) = \#\{\text{boxes to the right of } (i,j)\} + \#\{\text{boxes below } (i,j)\} + \text{the box } (i,j) \text{ itself.}$$

The number of semistandard Young tableaux of shape $\lambda \vdash_d n$ is equal to

$$m_{\lambda,d} = \prod_{1\le i<j\le d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$