Recap: Marginal state

State space of joint system AB**:** Tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Marginal state

The *marginal state* ρ_A of a bipartite state ρ_{AB} is defined as the operator ρ_A satisfying the following relation for all $X_A \in \mathcal{L}(\mathcal{H}_A)$:

$$\mathrm{tr}(\rho_{AB}(X_A \otimes \mathbb{1}_B)) = \mathrm{tr}(\rho_A X_A)$$

Uniquely defines *partial trace* $\mathrm{tr}_B = \mathrm{id}_A \otimes \mathrm{tr} : \mathcal{L}(\mathcal{H}_{AB}) \to \mathcal{L}(\mathcal{H}_A)$.

Partial trace in coordinates: For an orthonormal basis $\mathcal{E} = \{|e_i\rangle_B\}_{i=1}^{|B|}$,

$$\mathrm{tr}_B X_{AB} = \sum_{i=1}^{\dim B} \left(\mathbb{1}_A \otimes \langle e_i|_B\right) X_{AB} \left(\mathbb{1}_A \otimes |e_i\rangle_B\right).$$

Recap: Correlations

1. *Product states:* $\rho_{AB} = \omega_A \otimes \sigma_B$ for states ω_A and σ_B.
2. *Separable states:* $\rho_{AB} = \sum_i p_i \omega_A^{(i)} \otimes \sigma_B^{(i)}$ for states $\omega_A^{(i)}$ and $\sigma_B^{(i)}$ and prob. dist. $(p_i)_i$.
3. *Entangled states:* states that are not separable.

Examples of entangled states: *Bell states* or *maximally entangled states*

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\big) = \frac{1}{\sqrt{2}}(1,0,0,1)^T.$$

$$|\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B\big) = \frac{1}{\sqrt{2}}(1,0,0,-1)^T$$

$$|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\big) = \frac{1}{\sqrt{2}}(0,1,1,0)^T$$

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\big) = \frac{1}{\sqrt{2}}(0,1,-1,0)^T.$$

Recap: PPT criterion

We define the partial transpose as $(\cdot)^{T_B} := \mathrm{id}_A \otimes (\cdot)^T$.

PPT states

A state ρ_{AB} is called **PPT** (for *positive partial transpose*), if

$$\left(\rho_{AB}^{T_B} \geq 0 \iff \rho_{AB}^{T_A} \geq 0\right).$$

PPT criterion

Every separable state σ_{AB} satisfies $\sigma_{AB}^{T_B} \geq 0$. Hence, if $\rho_{AB}^{T_B}$ has a negative eigenvalue, then ρ_{AB} is entangled.

Recap: Representation of S_2

Maximally entangled states are indeed entangled by PPT criterion: $(\Phi^+)^{T_B} = \frac{1}{2}\mathbb{F} \ngeq 0$

$\mathbb{F}$ is called **swap operator** and represents the transposition $(12) \in S_2$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\mathbb{F}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle.$$

$\mathbb{F}$ acts trivially on Bell states $|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle$, but $\mathbb{F}|\Psi^-\rangle = -|\Psi^-\rangle$.

Symmetric subspace

$\mathrm{Sym}^2(\mathbb{C}^2) := \mathrm{span}(|\Phi^+\rangle_{AB}, |\Phi^-\rangle_{AB}, |\Psi^+\rangle_{AB})$ contains three copies of the 1-dimensional **trivial representation** of S_2.

Each $|\Phi^+\rangle_{AB}, |\Phi^-\rangle_{AB}, |\Psi^+\rangle_{AB}$ spans one copy, and both group elements $e, (12) \in S_2$ act trivially on each of them.

Antisymmetric subspace

$\Lambda^2(\mathbb{C}^2) := \mathrm{span}(|\Psi^-\rangle)$ is the 1-dimensional **sign representation** of S_2.

The identity e still acts trivially, transposition (12) acts by multiplying with -1.