# Admin

- Course website:
  `https://felixleditzky.info/teaching/FT25/math595_reptheory.html`
  (navigate there by going to `felixleditzky.info` → “Teaching”)
- Email: `leditzky@illinois.edu`
- Campuswire (by invitation, send me an email)

## Quantum Working Group Seminar

**When & Where:** Wednesdays, 4-5pm, 1022 Lincoln Hall (starting next week)

**Organizers:** Marius Junge, Felix Leditzky, Amanda Young

**Topics:** Quantum information, Operator algebras, ...

**Mailing list:** Subscribe by sending empty email with the following information:

To: `sympa@lists.illinois.edu`
Subject: `subscribe quantum-uiuc <first name> <last name>`

# Recap (Aug 26)

## Quantum systems & states

- A (finite-dim.) quantum system is associated with a Hilbert space $\mathcal{H} \cong \mathbb{C}^d$.
- A quantum state $\rho$ on $\mathcal{H}$ is a density operator, a linear operator that is
  - positive semidefinite: $\rho \geq 0 \quad (\Leftrightarrow \langle\psi|\rho|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$;
  - normalized: $\operatorname{tr}\rho = 1$.

## Pure vs. mixed states

- A pure state $\psi$ is a density operator of rank 1. It can be written as $\psi = |\psi\rangle\langle\psi|$ for some normalized vector $|\psi\rangle \in \mathcal{H}$.
- This is a slight abuse of notation, but very useful in practice!
- A mixed state is a density operator of rank $\geq 2$. It can always be written as a convex combination of pure states,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|,$$

  and there are infinitely many such ways to decompose $\rho$.

# Recap (Aug 26)

## Projective measurements

- An observable $A$ is a Hermitian operator on $\mathcal{H}$.
- Given the spectral decomposition $A = \sum_i a_i P_i$, the projective measurement corresponding to $A$ is given by the orthogonal projectors $\{P_i\}_i$ satisfying

$$\text{(i) } P_i \geq 0; \qquad \text{(ii) } \sum_i P_i = \mathbb{1}_{\mathcal{H}}; \qquad \text{(iii) } P_i P_j = \delta_{ij} P_i.$$

- The probability of obtaining measurement outcome $a_i$ is given by $p_i = \mathrm{tr}(\rho P_i)$.

## Positive operator-valued measurements (POVM)

- Property (iii) above is not needed to obtain measurement statistics.
- A POVM is a collection of operators $\{E_i\}$ satisfying

$$\text{(i) } E_i \geq 0 \qquad \text{(ii) } \sum_i E_i = \mathbb{1}_{\mathcal{H}}.$$

- The probability of obtaining measurement outcome "$i$" is given by $p_i = \mathrm{tr}(\rho E_i)$.

# Tensor product

Given two vector spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the **tensor product** $\mathcal{H}_A \otimes \mathcal{H}_B$ is a vector space with the following properties:

(i) For every $|\psi\rangle_A \in \mathcal{H}_A$ and $|\phi\rangle_B \in \mathcal{H}_B$ there is an associated **simple tensor** (for us: **product state**) $|\psi\rangle_A \otimes |\phi\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B$.
Every vector in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as a linear combination of such simple tensors.

(ii) The tensor product is bilinear: for every $|\psi_*\rangle_A \in \mathcal{H}_A$, $|\phi_*\rangle \in \mathcal{H}_B$ and $c \in \mathbb{C}$,

$$\begin{aligned} |\psi\rangle_A \otimes \left(|\phi_1\rangle_B + |\phi_2\rangle_B\right) &= |\psi\rangle_A \otimes |\phi_1\rangle_B + |\psi\rangle_A \otimes |\phi_2\rangle_B \\ \left(|\psi_1\rangle_A + |\psi_2\rangle_A\right) \otimes |\phi\rangle_B &= |\psi_1\rangle_A \otimes |\phi\rangle_B + |\psi_2\rangle_A \otimes |\phi\rangle_B \\ \left(c|\phi\rangle_A\right) \otimes |\phi\rangle_B &= |\psi\rangle_A \otimes \left(c|\phi\rangle_B\right) = c\left(|\psi\rangle_A \otimes |\phi\rangle_B\right). \end{aligned}$$

(iii) The tensor product is unique up to isomorphism (universal property).

We have $\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = (\dim\mathcal{H}_A) \cdot (\dim\mathcal{H}_B)$.

# Kronecker product

Tensor product “in coordinates” (after choosing bases).

**Example:** $\mathcal{H}_A = \mathbb{C}^2, \mathcal{H}_B = \mathbb{C}^3$.

$$|\psi\rangle_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, |\phi\rangle_B = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad \Longrightarrow \quad |\psi\rangle_A \otimes |\phi\rangle_B = \begin{pmatrix} \psi_1 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \\ \psi_2 \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \psi_1\phi_3 \\ \psi_2\phi_1 \\ \psi_2\phi_2 \\ \psi_2\phi_3 \end{pmatrix}$$

$$X_A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y_B = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

$$\Longrightarrow \quad X_A \otimes Y_B = \begin{pmatrix} x_{11} \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} & x_{12} \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \\ x_{21} \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} & x_{22} \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{11}y_{13} & x_{12}y_{11} & x_{12}y_{12} & x_{12}y_{13} \\ x_{11}y_{21} & x_{11}y_{22} & x_{11}y_{23} & x_{12}y_{21} & x_{12}y_{22} & x_{12}y_{23} \\ x_{11}y_{31} & x_{11}y_{32} & x_{11}y_{33} & x_{12}y_{31} & x_{12}y_{32} & x_{12}y_{33} \\ x_{21}y_{11} & x_{21}y_{12} & x_{21}y_{13} & x_{22}y_{11} & x_{22}y_{12} & x_{22}y_{13} \\ x_{21}y_{21} & x_{21}y_{22} & x_{21}y_{23} & x_{22}y_{21} & x_{22}y_{22} & x_{22}y_{23} \\ x_{21}y_{31} & x_{21}y_{32} & x_{21}y_{33} & x_{22}y_{31} & x_{22}y_{32} & x_{22}y_{33} \end{pmatrix}$$

Recommended literature: Nathaniel Johnston, “Advanced Linear and Matrix Algebra”