Consensus for quantum networks: symmetry from gossip interactions

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IEEE transactions on automatic control, 2015

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Outline

Part 1: Classical consensus and gossip protocol (**Chen-Wei**)

Part 2: Quantum consensus and gossip interactions (**Shashwat**)

Part 3: The emergence of symmetry (**Xiaojuan**)

Classical Consensus (Definition)

- Consider m subsystems with states $x_k \in \mathbb{R}^n$
    - The joint state is $x = (x_1, \ldots, x_m) \in \mathbb{R}^{mn}$
- Consensus state: all subsystems (agents) share the same state

$$\mathcal{C} = \{(x_1, \ldots, x_m) : x_j = x_k, \ \forall j, k\}$$

- Average consensus: given initial states $x_i(0)$, they converge to

$$(\bar{x}, \ldots, \bar{x}) \in \mathcal{C}, \quad \bar{x} = \frac{1}{m} \sum_i x_i(0)$$

Classical Consensus

Centralized Approach:

- All nodes send their initial values to a central node *C*
- *C* computes, and sends back to all nodes.

Issues:

1. High Communication Overhead
2. Single point of failure
3. Poor Adaptability to Dynamic Networks

Distributed Approach:

- Compute the average in a distributed way. I.e., gossip protocol.

Locality: Quasi-Local operators

- A quasi-local operator: $\mathcal{N}_j \subseteq \{1, \ldots, m\}, j = 1, \ldots, M$
  - Dynamics with local coupling can be written as: $\mathcal{N}_j$
  - leaves all other agents unchanged
- Dynamics with local coupling can be written as:

$$x(t+1) = \sum_{j=1}^{M} V_j(x(t))$$

where each $V_j : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ is quasi-local on $\mathcal{N}_j$

Distributed computation and Locality

- Each agent k has a state $x_k(t)$
- First–order integrator dynamics: discrete time case in this paper

$$x_k(t+1) = x_k(t) + u_k(t)$$

- Control input $u_k(t)$:
  - may have different forms
  - is based only on local information
- Locality modeled by a graph $G(V, E)$:
  - Vertices $1, \ldots, m$ represent the agents
  - Edge $(j, k) \in E$ means agents j and k are neighbors

Distributed computation and Locality

- For each k, input $u_k(t)$ depends only on:
  - its own state $x_k(t)$
  - states $x_j(t)$ of neighboring agents j with $(j,k) \in E$
- Often written as a sum of contributions from edges:

$$u_k(t) = \sum_{j:(j,k)\in E} f_{j,k}(x_k(t), x_j(t))$$

Timing of operations: how to select edges

In classical consensus, the interaction graph may change over time and edges may be active or inactive at different time steps.

- **Synchronous** update: all edges are active at every time
- **Asynchronous** update: only one edge, or a subset of edges, is active at each time
- The active edges can be chosen:
  - by a predefined time-varying schedule, or
  - by random selection of edges

Randomized gossip protocol

- At each iteration:
  - Randomly pick one edge based some predefined probability distribution
  - all others stay the same
- Update rule: move towards each other / their mean

$$x_j(t+1) = x_j(t) + \alpha\big(x_k(t) - x_j(t)\big)$$

$$x_k(t+1) = x_k(t) + \alpha\big(x_j(t) - x_k(t)\big)$$

where $\alpha \in (0, 1)$

$$x_\ell(t+1) = x_\ell(t) \quad \text{for all } \ell \notin \{j, k\}$$

Alternative view of the interaction

- interacting agents take a weighted average between [keep st] and [swap st]
- The “keep / swap” viewpoint has a natural quantum counterpart

$$(x_j(t+1),\, x_k(t+1)) = (1-\alpha)\,(x_j(t),\, x_k(t)) + \alpha\,(x_k(t),\, x_j(t)),$$

$$x_\ell(t+1) = x_\ell(t) \qquad \text{for all } \ell \notin \{j,k\}.$$

Convergence state: $(\bar{x},\dots,\bar{x}) \in \mathcal{C}, \quad \bar{x} = \frac{1}{m}\sum_i x_i(0)$

“Quantum Consensus”

- A classical consensus state for a multipartite system is one in which all the states of all subsystems are the same.
- For a quantum network, we try to draw analogies from the classical case.
- However, as quantum measurement outcomes are inherently stochastic, we must consider probabilistic consensus situations.

We need a few fundamental things

- How to define consensus?
- How do subsystems interact (gossip) with each other?
- What is locality?
- How does the system evolve over time?

When is a Quantum Network in Consensus?

- Consider a multipartite quantum system composed of 3 qubits
- Associated Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
- 3 observables $\sigma^{(1)} = \sigma_z \otimes I \otimes I$, $\sigma^{(2)} = I \otimes \sigma_z \otimes I$, $\sigma^{(3)} = I \otimes I \otimes \sigma_z$

where the Pauli matrix $\sigma_z = \mathbf{diag}(1, -1)$

- Condition for consensus:

$$\mathrm{Tr}(\rho\sigma^{(1)}) = \mathrm{Tr}(\rho\sigma^{(2)}) = \mathrm{Tr}(\rho\sigma^{(3)})$$

σ-Expectation Consensus

More generally, observables,

$$\sigma^{(i)} := I^{\otimes(i-1)} \otimes \sigma \otimes I^{\otimes(m-i)}$$

Definition 1 (σEC): Given $\sigma \in \mathfrak{B}(\mathcal{H})$, a state $\rho \in \mathfrak{D}(\mathcal{H}^m)$ is in *σ-Expectation Consensus* (σEC) if:

$$\mathrm{Tr}(\sigma^{(1)}\rho) = \ldots = \mathrm{Tr}(\sigma^{(k)}\rho).$$

A few examples

$$\begin{aligned}
\rho^A &= \frac{1}{8} I \otimes (|0\rangle + |1\rangle)(\langle 0| + \langle 1|) \otimes (|0\rangle + |1\rangle)(\langle 0| + \langle 1|); \\
\rho^B &= \frac{1}{4} I \otimes (|0,0\rangle + |1,1\rangle)(\langle 0,0| + \langle 1,1|); \\
\rho^C &= \frac{1}{8} I \otimes I \otimes I; \\
\rho^D &= \frac{1}{2}(|0,0,0\rangle\langle 0,0,0| + |1,1,1\rangle\langle 1,1,1|); \\
\rho^E &= |0,0,0\rangle\langle 0,0,0|; \\
\rho^F &= \frac{1}{2}(|0,0,0\rangle + |1,1,1\rangle)(\langle 0,0,0| + \langle 1,1,1|).
\end{aligned}$$

All these states, except ρ^E, have $\mathrm{Tr}(\rho\sigma^{(i)}) = 0$ for $i = 1, 2, 3$.

Reduced State Consensus

- We want to be able to ascertain consensus using any observable
- Replacing σ_z with *any* observable $\sigma \in \mathfrak{B}(\mathbb{C}^2)$
- This is equivalent to imposing that the reduced states for the three subsystems are the same.
- All expect ρ^A satisfy this requirement $\rho_1^A = \frac{1}{2}I, \quad \rho_2^A = \rho_3^A = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$
- More generally

Definition 2 (RSC): A state $\rho \in \mathfrak{D}(\mathcal{H}^m)$ is in *Reduced State Consensus* (RSC) if, defining the reduced states $\bar{\rho}_k = \mathrm{Tr}_{(\bigotimes_{j \neq k} \mathcal{H}_j)}(\rho)$, we have

$$\bar{\rho}_1 = \ldots = \bar{\rho}_m.$$

Symmetric State Consensus

- Classical consensus is invariant under any permutations of the subsystems
- As a consequence, another potential definition of quantum consensus would require the state to be symmetric
- Out of the examples, only $\rho^C, \rho^D, \rho^E, \rho^F$ are permutation invariant.
- Permutations of quantum subsystems are expressed by $U_\pi \in \mathfrak{U}(\mathcal{H})$

$$U_\pi(X_1 \otimes \ldots \otimes X_m)U_\pi^\dagger = X_{\pi(1)} \otimes \ldots \otimes X_{\pi(m)}$$

- More generally,

Definition 3 (SSC): A state $\rho \in \mathfrak{D}(\mathcal{H}^m)$ is in *Symmetric State Consensus* (SSC) if for each unitary permutation U_π we have

$$U_\pi \, \rho \, U_\pi^\dagger = \rho$$

Overview

The following chain of implication holds $\mathbf{SSC} \implies \mathbf{RSC} \implies \sigma\mathbf{EC}$ while the converse implications are not true in general.

Proof: SSC $\implies$ *RSC*: If $U_\pi \rho U_\pi^\dagger = \rho$ for each permutation, consider in particular $U_{(\ell,k)}$ that swaps subsystems ℓ and k. Then

$$\bar{\rho}_k = \mathrm{Tr}_{\bigotimes_{j\neq k}\mathcal{H}_j}(\rho) = \mathrm{Tr}_{\bigotimes_{j\neq k}\mathcal{H}_j}(U_{(\ell,k)}\rho U_{(\ell,k)}^\dagger) = \bar{\rho}_\ell,$$

and the reasoning can be repeated for any pair. *RSC* $\implies$ *OSC* is immediate by definition. States ρ^B and ρ^A from Example 1 provide counterexamples for the converse of the first and of the second implication, respectively. □

Evolution in Quantum Network

- According to Schrodinger's equation, closed quantum systems evolve unitarily.
- However, unitary dynamics are not enough when studying convergence.
- We need a more general framework that includes open system evolutions.
- Quantum channels (linear, completely positive (CP) and trace preserving (TP) maps)
- It can be shown that such maps admit an operator sum representation (OSR), also known as Kraus decomposition.

$$\mathcal{E}(\rho) = \sum_{k=1}^{K} A_k \rho A_k^\dagger \quad \textbf{with} \quad \sum_{k=1}^{K} A_k^\dagger A_k = I$$

Locality in Quantum Network

- We also need locality notions for the quantum network.
- We say that an operator is quasi-local if it acts non-trivially only on one neighborhood
- We say that an operator in $\mathfrak{B}(\mathcal{H})$ is quasi-local if it satisfies the following definition.

Definition 5 (Quantum quasi-local operator): An operator V is quasi-local with respect to a set of neighborhoods $\{\mathcal{N}_j \ , j = 1, 2, ..., M\}$, if and only if there exists $j \in \{1, 2, ..., M\}$ such that:

$$V = V_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}_j}} \tag{13}$$

Gossip Interactions

- The interacting agents take a weighted average between two discrete operations
  - Keep your state
  - Swap your state

$$
\begin{aligned}
(x_j(t+1),\ x_k(t+1)) &= (1-\alpha)\ (x_j(t),\ x_k(t)) \\
&\quad + \alpha\ (x_k(t),\ x_j(t)) \\
x_\ell(t+1) &= x_\ell(t) \qquad \text{for all } \ell \notin \{j,k\}
\end{aligned}
$$

- Quantum gossip interaction implementing the quantum channel

$$
\rho(t+1) = \mathcal{E}_{j,k}(\rho(t)) = (1-\alpha)\,\rho(t) + \alpha\, U_{(j,k)}\rho(t)U_{(j,k)}^{\dagger}
$$

Asymptotic consensus

The convex set of density operators

Let $d(\rho_a, \mathcal{C}) = \inf_{\rho \in \mathcal{C}} \|\rho_a - \rho\|$, where $\mathcal{C} \subset \mathfrak{D}(\mathcal{H})$ and $\|\cdot\|$ is any p-norm on $\mathfrak{B}(\mathcal{H})$. Given a sequence of channels $\{\mathcal{E}_t(\cdot)\}_{t=0}^{\infty}$, define $\hat{\mathcal{E}}_t(\rho_0) = \rho_t = \mathcal{E}_t \circ \mathcal{E}_{t-1} \circ \cdots \circ \mathcal{E}_1(\rho_0)$, and $\mathcal{C}_{\sigma\mathrm{EC}}$ to be the set of states in σEC consensus.

Definition 6 (Asymptotic Consensus): A sequence of channels $\{\mathcal{E}_t(\cdot)\}_{t=0}^{\infty}$, is said to *asymptotically achieve σEC* if

$$\lim_{t\to\infty} d\left(\hat{\mathcal{E}}_t(\rho_0), \mathcal{C}_{\sigma\mathrm{EC}}\right) = 0 \tag{16}$$

for all initial states ρ_0.

Same def holds for RSC and SSC with corresponding state sets.

$$\mathcal{E}(X) = q_0 X + \sum_{(j,k)\in E} q_{j,k} U^\dagger_{(j,k)} X U_{(j,k)}$$

$$\text{with } q_0 + \sum_{(j,k)\in E} q_{j,k} = 1, q_0, \{q_{j,k}\} > 0$$

Quantum gossip: identity and swap: generate entire Sn

$$\rho_* \stackrel{?}{=} \frac{1}{m!} \sum_{\pi\in\mathfrak{P}} U_\pi \rho_0 U_\pi^\dagger$$

$$\mathcal{E}(X) = q_0 X + \sum_{(j,k)\in E} q_{j,k} U^\dagger_{(j,k)} X U_{(j,k)}$$

self-adjoint

$$\text{with } q_0 + \sum_{(j,k)\in E} q_{j,k} = 1, q_0, \{q_{j,k}\} > 0$$

- The channel is self-adjoint $\mathcal{E} = \mathcal{E}^\dagger$
- Diagonalizable : eigenvectors form orthogonal basis
- Any initial state can be expressed in the basis $\rho_0 = \sum_i c_i X_i$
- Different modes evolve independently according to the eigenvalues

$$\mathcal{E}^n(\rho_0) = \sum_i c_i \mathcal{E}^n(X_i) = \sum_i c_i \lambda_i^n X_i$$

$$\mathcal{E}^n(\rho_0) = \sum_i c_i \mathcal{E}^n(X_i) = \sum_i c_i \lambda_i^n X_i$$

All modes with $|\lambda| < 1$ decay exponentially.

Modes with $|\lambda| = 1$ remain.

- $|\lambda| = 1, \lambda \neq 1$ (Oscillatory/Limit Cycle)

  Example: two qubits system with SWAP channel

- $\lambda = 1$ (Fixed Points) $\mathrm{Fix}(\mathcal{E}) = \{X \in \mathrm{End}(\mathcal{H}) \mid \mathcal{E}(X) = X\}$

  If converge, the limit state is a fixed point of the channel.

Will show:

- $|\lambda| \leq 1$ for all CPTP maps
- $\lambda = 1$ exists for all CPTP maps
- If $|\lambda| = 1$ then $\lambda = 1$ for our quantum gossip channel: converge.

Eigenvalues of a CPTP map: $|\lambda| \leq 1$

1. $|\lambda| \leq 1$: Trace norm contraction of CPTP maps

$$||A||_{\mathrm{tr}} := \mathrm{Tr}(\sqrt{A^\dagger A}) = \sum_i s_i(A)$$

s_i: Singular values

$$||\mathcal{E}(A) - \mathcal{E}(B)||_{\mathrm{tr}} \leq ||A - B||_{\mathrm{tr}}$$

Quantum channel cannot improve distinguishability

$$||\mathcal{E}(A)||_{\mathrm{tr}} \leq ||A||_{\mathrm{tr}}$$

Set B = 0

$$||\lambda A||_{\mathrm{tr}} \leq ||A||_{\mathrm{tr}}$$

A: Eigenstate

$$|\lambda| \leq 1$$

λ=1 is an eigenvalue for any CPTP map

2. Every CPTP map has at least one fixed point: Trace-preserving

- Schrödinger Picture (State Evolution): operator is static $\rho_{\text{out}} = \mathcal{E}(\rho_{\text{in}})$
- Heisenberg Picture (Operator Evolution): state is static $A_{\text{out}} = \mathcal{E}^\dagger(A_{\text{in}})$
- $\mathcal{E}$ is TP $\iff$ $\mathcal{E}^\dagger$ is Unital

$$\mathcal{E}^\dagger(I) = I$$

λ=1

- $\mathcal{E}$ and $\mathcal{E}^\dagger$ share the same spectrum (set of eigenvalues)

Quantum gossip: if $|\lambda| = 1$, then $\lambda = 1$.

Lemma 2: Consider a linear completely positive map $\mathcal{E}$ on $\mathfrak{B}(\mathcal{H})$ that admits an operator-sum representation $\{A_k\}$ with one operator proportional to identity, i.e., $A_1 = \sqrt{\alpha} I > 0$. Then, if λ is an eigenvalue of $\mathcal{E}$, $|\lambda| = 1$ implies $\lambda = 1$.

Example: two-qubit system

$$\mathcal{E} = q_0 \mathcal{I} + (1 - q_0)\mathcal{E}_{\text{swap}},$$

$$\mathcal{E}(X) = q_0 \mathcal{I}(X) + (1 - q_0)\mathcal{U}_{\text{swap}}(X)$$

$$\lambda_{\mathcal{E}} = q_0 \lambda_{\mathcal{I}} + (1 - q_0)\lambda_{\mathcal{U}_{\text{swap}}}$$

$$\lambda_{\text{anti}} = q_0 \cdot (1) + (1 - q_0) \cdot (-1) \qquad |\lambda_{\text{anti}}| = |2q_0 - 1| < 1$$

Proof: further limit the possible values of eigenvalues

1. $\mathcal{E}$ Eigenvalues λ_k : within closed unit disk.
2. Construct another CPTP map $\mathcal{F}$ with μ_k : again, unit circle

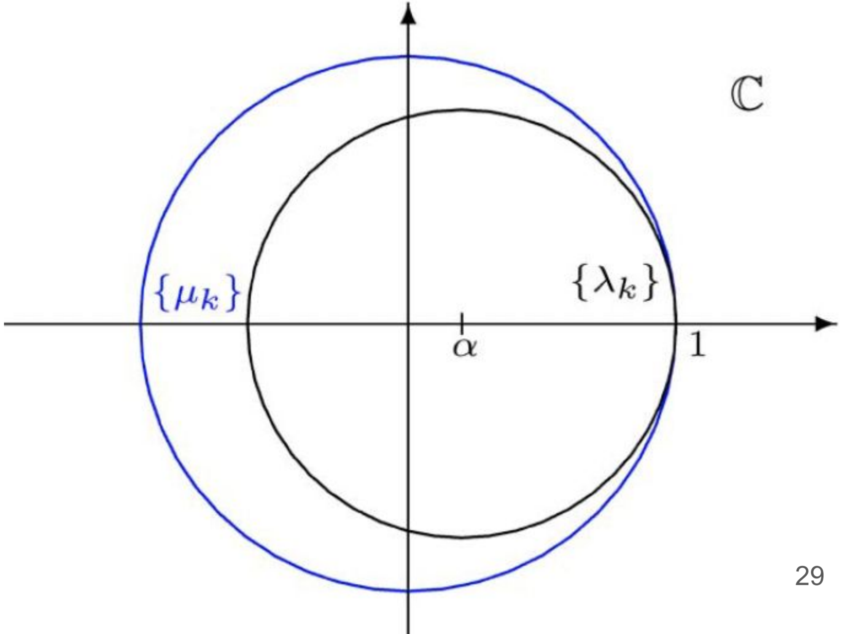
$$\mathcal{F} = (1/(1-\alpha))(\mathcal{E} - \alpha I)$$

3. $$\mathcal{E} = (1-\alpha)\mathcal{F} + \alpha I$$

$$\lambda_k = (1-\alpha)\mu_k + \alpha$$

4. λ_k touches the unit circle only at 1

Quantum gossip protocol converges.



What are fixed points of QGP?

Proposition 4: Let $\{V_i\}_{i=1}^{K}$ the Kraus decomposition of a *unital* CP map $\mathcal{E}(\cdot)$ and define

$$\mathcal{A}_{\mathcal{E}} = \{X \in \mathfrak{B}(\mathcal{H}^m) | XV_i - V_i X = 0 \ \forall\, i = 1, \ldots, K\}. \quad (22)$$

Then $\bar{X} \in \mathfrak{B}(\mathcal{H}^m)$ is a fixed point of $\mathcal{E}$, i.e., $\mathcal{E}(\bar{X}) = \bar{X}$, if and only if $\bar{X} \in \mathcal{A}_{\mathcal{E}}$. ■

Quantum gossip: identity and swap:

- unital
- generate entire Sn

Fixed points of quantum gossip:

- permutation-invariant operators

QGC: Preserves the expectation value of permutation invariant observables

S: permutation invariant observable

$$S = U_{(j,k)} S U^\dagger_{(j,k)} = U_{(j,k)} S U_{(j,k)} = U^\dagger_{(j,k)} S U_{(j,k)}$$

Def of permutation invariant ob

Evolve S: Permutation invariant observables are fixed points of the dual map

$$U_{(j,k)} S U^\dagger_{(j,k)} = S \Rightarrow \mathrm{Tr}\left[\mathcal{E}_{j,k}(\rho) S\right] = \mathrm{Tr}\left[\rho \mathcal{E}^\dagger_{j,k}(S)\right] = \mathrm{Tr}[\rho S]$$

Evolve state, apply S

Evolve S, Equals to S

Preserves the expectation value of S

Q: arbitrary observable

$$X = \frac{1}{m!} \sum_{\pi \in \mathfrak{P}} U_\pi^\dagger Q U_\pi \qquad \mathrm{Tr}\left[X \mathcal{E}_C(\rho_0)\right] = \mathrm{Tr}[X \rho_0]$$

ρ_∞ Permutation invariant

Cyclic + U is self-adjoint;
Constructed a X

$$\mathrm{Tr}[Q\rho_\infty] = \mathrm{Tr}\left[Q \frac{1}{m!} \sum_{\pi \in \mathfrak{P}} U_\pi \rho_\infty U_\pi^\dagger\right] = \mathrm{Tr}\left[\frac{1}{m!} \sum_{\pi \in \mathfrak{P}} U_\pi Q U_\pi^\dagger \rho_\infty\right]$$

$$= \mathrm{Tr}\left[\frac{1}{m!} \sum_{\pi \in \mathfrak{P}} U_\pi Q U_\pi^\dagger \rho_0\right] = \mathrm{Tr}\left[\frac{1}{m!} \sum_{\pi \in \mathfrak{P}} Q U_\pi \rho_0 U_\pi^\dagger\right].$$

$$\rho_* = \frac{1}{m!} \sum_{\pi \in \mathfrak{P}} U_\pi \rho_0 U_\pi^\dagger$$

$\mathrm{Tr}\left[X \mathcal{E}_C(\rho_0)\right] = \mathrm{Tr}[X \rho_0]$

Cyclic + U self-adjoint

Locally accessible global info: S-average consensus

Definition 7 (Asymptotic Average Consensus): We say that the sequence of channels $\{\mathcal{E}_t(\cdot)\}_{t=0}^{\infty}$ asymptotically achieves *S-average σEC* for some $S \in \mathfrak{H}(\mathcal{H}^m)$ if it asymptotically achieves σEC and for all ρ_0, it holds

$$\lim_{t\to\infty} \mathrm{Tr}\left(\sigma \bar{\rho}_\ell(t)\right) = \lim_{t\to\infty} \mathrm{Tr}\left(\sigma^{(\ell)} \rho(t)\right) = \lim_{t\to\infty} \mathrm{Tr}\left(S \rho(t)\right)$$
$$= \mathrm{Tr}(S\rho_0) \quad (17)$$

local global global

for all $\ell \in \{1, \ldots, m\}$.

Proposition 2: Consider a sequence of CPTP channels $\{\mathcal{E}_t(\cdot)\}_{t=0}^{\infty}$, and call $\hat{\mathcal{E}}_t = \mathcal{E}_t \circ \mathcal{E}_{t-1} \circ \ldots \circ \mathcal{E}_1$. The associated dynamics satisfies (17) if and only if

$$S = \lim_{t\to\infty} \hat{\mathcal{E}}_t^{\dagger}(S) \quad \text{and} \quad \lim_{t\to\infty} \hat{\mathcal{E}}_t^{\dagger}\left(\sigma^{(\ell)}\right) = S \qquad (18)$$

for $\ell = 1, 2, \ldots, m$, where $\hat{\mathcal{E}}_t^{\dagger} = \mathcal{E}_1^{\dagger} \circ \mathcal{E}_2^{\dagger} \circ \ldots \circ \mathcal{E}_t^{\dagger}$.

Claim: $$S = \frac{1}{m}\sum_{i=1}^{m} \sigma^{(i)}$$

Claim: $S = \frac{1}{m}\sum_{i=1}^{m}\sigma^{(i)}$

To show :

$$\lim_{t\to\infty}\hat{\mathcal{E}}_t^\dagger(\sigma^{(\ell)}) = S$$

$$\lim_{t\to\infty}\hat{\mathcal{E}}_t^\dagger(\sigma^{(\ell)}) = \frac{1}{m!}\sum_{\pi\in\mathfrak{P}} U_\pi^\dagger \sigma^{(\ell)} U_\pi = \frac{1}{m}\sum_{i=1}^{m}\sigma^{(i)}. \qquad (29)$$

A toy example: $S = \frac{1}{4}\left(\sigma_z^{(1)} + \sigma_z^{(2)} + \sigma_z^{(3)} + \sigma_z^{(4)}\right)$ $\rho = |1,0,1,0\rangle\langle 1,0,1,0|$

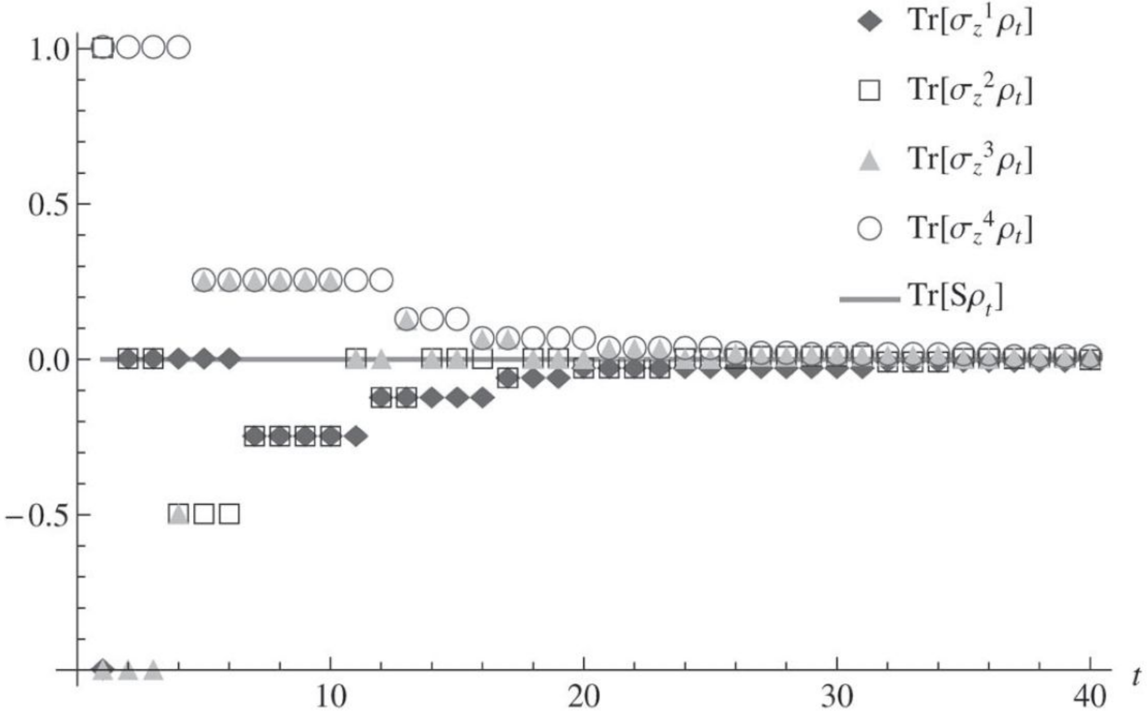


Fig. 2. Evolution toward σ-Expectation Consensus for a four-qubit network arranged in a path graph.