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Math 595 Representation-theoretic methods in QIT

Spectrum estimation

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Problem setup

Spectrum estimation

In this section we are interested in the task of estimating the spectrum (i.e., the set of eigenvalues) of an unknown density operator ρ of a quantum system.

Assumptions

- (i) We have access to an experiment that prepares the system (exactly) in the state ρ .
- (ii) We can run this experiment n times and perform joint measurements on all n copies at the same time. The goal is to estimate the spectrum of ρ by making a suitable measurement on $\rho^{\otimes n}$.

The goal is to devise a strategy that correctly estimates the true spectrum of ρ with probability approaching 1 as $n \rightarrow \infty$.

Symmetries of spectrum estimation

Goal: Solve spectrum estimation by exploiting the symmetries of this problem.

Permutation invariance

$\rho^{\otimes n}$ is permutation invariant, $Q_\pi \rho^{\otimes n} Q_\pi^\dagger = \rho^{\otimes n}$ for all $\pi \in S_n$.

Let now $P \geq 0$ be an arbitrary measurement operator. For all $\pi \in S_n$,

$$\text{tr}(P\rho^{\otimes n}) = \text{tr}(PQ_\pi \rho^{\otimes n} Q_\pi^\dagger) = \text{tr}(Q_\pi^\dagger P Q_\pi \rho^{\otimes n}),$$

and so $\text{tr}(P\rho^{\otimes n}) = \text{tr}(\bar{P}\rho^{\otimes n})$ for the symmetrized effect operator \bar{P} defined as

$$\bar{P} = \frac{1}{n!} \sum_{\pi \in S_n} Q_\pi P Q_\pi^\dagger.$$

We can do this for all elements $M_i \geq 0$ of a POVM, and the symmetrized operators \bar{M}_i still form a POVM:

$$\sum_i \bar{M}_i = \sum_i \frac{1}{n!} \sum_{\pi \in S_n} Q_\pi M_i Q_\pi^\dagger = \frac{1}{n!} \sum_{\pi \in S_n} Q_\pi \left(\sum_i M_i \right) Q_\pi^\dagger = \mathbb{1}.$$

Symmetries of spectrum estimation

Unitary covariance

The states ρ and $U\rho U^\dagger$ have the same eigenvalues for any unitary $U \in \mathcal{U}_d$.

Thus, any measurement performed on $\rho^{\otimes n}$ that is supposed to estimate the spectrum of ρ should be invariant under $U^{\otimes n}$ as well.

We therefore *assume* for the time being this $U^{\otimes n}$ -invariance for the measurement, but one can prove that this is also without loss of generality [MW16, Lem. 20].

Schur-Weyl decomposition: $(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} V_\lambda \otimes U_\lambda^d$.

Denote by P_λ the projection onto the λ -isotypical component $V_\lambda \otimes U_\lambda^d$:

- (i) $P_\lambda \geq 0$ and $\sum_{\lambda \vdash_d n} P_\lambda = \mathbb{1}$.
- (ii) $[P_\lambda, Q_\pi] = 0$ for all $\pi \in S_n$.
- (iii) $[P_\lambda, U^{\otimes n}] = 0$ for all $U \in \mathcal{U}_d$.

Weak Schur sampling

$\{P_\lambda\}_{\lambda \vdash_d n}$ is a bona fide measurement called **weak Schur sampling**.

What does the outcome ' $\lambda \vdash_d n$ ' mean?

Partition $\lambda = (\lambda_1, \dots, \lambda_d) \vdash_d n$ satisfies $\lambda_1 \geq \dots \geq \lambda_d \geq 0$ and $\sum_{i=1}^d \lambda_i = n$.

Therefore, the normalized partition

$$\bar{\lambda} = \frac{1}{n}\lambda = (\lambda_1/n, \dots, \lambda_d/n)$$

satisfies $\bar{\lambda}_i \geq 0$ and $\sum_{i=1}^d \bar{\lambda}_i = 1$, and is thus a valid (ordered) spectrum.

Spectrum estimation protocol [ARS88; KW01]

Let ρ have spectrum $r = (r_1, \dots, r_d)$, and assume w.l.o.g. that $r_1 \geq r_2 \geq \dots \geq r_d \geq 0$.

- (i) Measure $\rho^{\otimes n}$ using the measurement $\{P_\lambda\}_{\lambda \vdash_d n}$.
- (ii) Upon obtaining outcome $\lambda \vdash_d n$, we set $\hat{r} = \frac{\lambda}{n}$.

We will prove that $\text{Prob}(\hat{r} \neq r) \rightarrow 0$ as $n \rightarrow \infty$.

Weak Schur sampling

Majorization

Goal: bound the probability $\text{tr}(P_\lambda \rho^{\otimes n})$ of obtaining outcome “ λ ” in weak Schur sampling. We will use the concept of *majorization* [MOA11] for this.

Majorization

Let $x, y \in \mathbb{R}^d$, and denote by $x^\downarrow, y^\downarrow$ the vectors of components of x, y sorted in non-increasing order (e.g. $x_1^\downarrow \geq \dots \geq x_d^\downarrow$). Then y is said to *majorize* x , in symbols $x < y$ if

$$\sum_{i=1}^q x_i^\downarrow \leq \sum_{i=1}^q y_i^\downarrow \text{ for all } q = 1, \dots, d-1, \text{ and } \sum_{i=1}^d x_i = \sum_{i=1}^d y_i.$$

Permutation invariance of $\rho^{\otimes n}$ and Schur-Weyl duality:

$$\rho^{\otimes n} = \bigoplus_{\lambda \vdash d^n} \mathbb{1}_{V_\lambda} \otimes \rho_\lambda$$

for some positive semidefinite operators $\rho_\lambda \in \text{End}(U_\lambda^d)$.

With the spectral decomposition $\rho = \sum_{i=1}^d r_i |e_i\rangle\langle e_i|$, form the tensor product basis

$$B = \left\{ \bigotimes_{j=1}^n |e_{i_j}\rangle : i_j \in [d] \right\}$$

For $|v\rangle \in B$ let $f = (f_1, \dots, f_d)$ be the *frequency distribution* of $|v\rangle$, where f_i is the number of times $|e_i\rangle$ appears in $|v\rangle$. Note that f is an (ordered) partition of n .

Lemma

Let $|v\rangle \in B$ with frequency distribution f , and let T be the standard Young tableau of shape $\lambda \vdash_d n$. Then $e_T |v\rangle = 0$ unless $f < \lambda$.

Proof in lecture notes and given in lecture.

We state our bounds on the weak Schur sampling probabilities $\text{tr}(P_\lambda \rho^{\otimes n})$ in terms of the *relative entropy* (also called *Kullback-Leibler divergence*) between two probability distributions p, q over the same alphabet, defined as

$$D(p\|q) = \begin{cases} \sum_i p_i \log \frac{p_i}{q_i} & \text{if } \text{supp}(p) := \{i : p_i \neq 0\} \subseteq \text{supp}(q); \\ \infty & \text{otherwise.} \end{cases}$$

The relative entropy satisfies $D(p\|q) \geq 0$ and $D(p\|q) = 0$ if and only if $p = q$, and hence we can view it as a non-symmetric “distance measure” on probability distributions.

Bound on weak Schur sampling probabilities

The main result of this section states that the probability of obtaining outcome “ λ ” in weak Schur sampling is exponentially small if $\bar{\lambda}$ is far from the spectrum of ρ in relative entropy distance.

Weak Schur sampling

Let ρ be a density operator with spectrum $r = (r_1, \dots, r_d)$ where $r_1 \geq r_2 \geq \dots \geq r_d \geq 0$, let $\lambda = (\lambda_1, \dots, \lambda_d) \vdash_d n$ and set $\bar{\lambda} = \frac{\lambda}{n}$. Then,

$$\text{tr}(P_\lambda \rho^{\otimes n}) \leq (n+1)^{\frac{d(d-1)}{2}} \exp(-nD(\bar{\lambda} \| r)).$$

Proof of the proposition

Proof in lecture notes and given in lecture.

Asymptotics of spectrum estimation

Testing a set of spectra

Previous section: for a quantum state ρ with spectrum $r = (r_1, \dots, r_d)$ and $\lambda \vdash_d n$,

$$\text{tr}(P_\lambda \rho^{\otimes n}) \leq (n+1)^{\frac{d(d-1)}{2}} \exp(-nD(\bar{\lambda} \| r)),$$

where $\bar{\lambda} = \lambda/n$ and $D(\cdot \| \cdot)$ is the relative entropy.

Extend this bound to a set S of possible spectra by defining

$$P_S = \sum_{\substack{\lambda \vdash n, \\ \bar{\lambda} \in S}} P_\lambda.$$

Probability bound

$$\text{tr}(P_S \rho^{\otimes n}) \leq (n+1)^{\frac{d(d-1)}{2}} \exp\left(-n \min\{D(\bar{\lambda} \| r) : \lambda \vdash n, \bar{\lambda} \in S\}\right)$$

Testing a set of spectra

Probability bound

$$\text{tr}(P_S \rho^{\otimes n}) \leq (n+1)^{\frac{d(d-1)}{2}} \exp\left(-n \min\{D(\bar{\lambda} \| r) : \lambda \vdash n, \bar{\lambda} \in S\}\right)$$

Proof.

This bound follows from picking the λ with the slowest convergence in S , or equivalently the minimum relative entropy distance $D(\bar{\lambda} \| r)$ from the true spectrum, and using the (rough, but sufficient) bound

$$|S| \leq |\{\lambda \vdash_d n\}| \leq (n+1)^d. \quad \square$$

For fixed $\varepsilon > 0$ we consider the ε -ball

$$B_\varepsilon(r) = \left\{ r' : \sum_i |r_i - r'_i| < \varepsilon \right\}$$

around the true spectrum r .

Asymptotic spectrum estimation

Choosing $S = (B_\varepsilon(r))^c$, the complement of $B_\varepsilon(r)$, we obtain:

Asymptotic spectrum estimation

Let ρ be a quantum state with (ordered) spectrum $r = (r_1, \dots, r_d)$, and for given $\varepsilon > 0$ let

$$P_X = \sum_{\substack{\lambda \neq r \\ \bar{\lambda} \in B_\varepsilon(r)}} P_\lambda.$$

Then for any $\delta > 0$ there exists n_0 such that for all $n \geq n_0$ we have $\text{tr}(P_X \rho^{\otimes n}) \geq 1 - \delta$.

Recall the total variation distance of two probability vectors $p, q \in \mathbb{R}^d$ defined as

$$d_{\text{TV}}(p, q) := \frac{1}{2} \sum_{j=1}^d |p_j - q_j|.$$

Pinsker's inequality:

$$d_{\text{TV}}(p, q) \leq \sqrt{\frac{1}{2} D(p \| q)}.$$

Sample complexity of spectrum estimation

Choosing $S = \{\lambda \vdash_d n : d_{\text{TV}}(\bar{\lambda}, r) > \varepsilon\}$ gives the following statement:

Sample complexity of spectrum estimation; [OW15]

For a mixed state ρ with ordered spectrum $r = (r_1, \dots, r_d)$, we have for any $\varepsilon > 0$ that

$$\Pr \left[d_{\text{TV}}(\bar{\lambda}, r) > \varepsilon \right] \leq (n + 1)^{d(d+1)/2} \exp(-2n\varepsilon^2).$$

Hence, $O(d^2/\varepsilon^2)\log(d/\varepsilon)\log(1/\delta)$ samples are sufficient to output an estimate $\bar{\lambda}$ for the spectrum of ρ satisfying $d_{\text{TV}}(\bar{\lambda}, r) \leq \varepsilon$ with probability at least $1 - \delta$.

- Best known sample complexity: $O(d^2/\varepsilon^2)$
- $\Omega(d/\varepsilon^2)$ copies are needed for spectrum estimation [OW15; Wri16].
- $O(d^2/\varepsilon^2)$ is also the scaling of full state tomography.
- **Open question:** Prove matching bounds [Wri16].
- Recent result [Pel+25]: separation when only unentangled measurements are considered.

References

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