Math 595 Representation-theoretic methods in QIT

The de Finetti theorem

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Extendibility of quantum states

Extendibility of quantum states

Definition

A bipartite state ρ_{AB} is called k-extendible if there exists a state $\rho_{AB_1\cdots B_k}$ (called k-extension) where each $B_i \cong B$ is a copy of the B-system and

$$\rho_{AB_i} = \operatorname{tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_k}\rho_{AB_1\cdots B_k} = \rho_{AB} \quad \text{for all } i = 1,\ldots,k.$$

Note: Some authors require the k-extension to be invariant under permuting the B-systems (which can be done without loss of generality).

Intuitively, in the k-extension $\rho_{AB_1\cdots B_k}$ of a k-extendible state ρ_{AB} the A system is simultaneously entangled to the same degree with each of the B_i-systems.

Pure entangled states are not even 2-extendible

Any state $\rho_{ABB'}$ satisfying $\operatorname{tr}_{B'}\rho_{ABB'} = \psi_{AB}$ is of the form $\rho_{ABB'} = \Phi^+_{AB} \otimes \omega_{B'}$ for some state $\omega_{B'}$ (exercise). Thus, if ψ_{AB} is entangled, then clearly

$$\operatorname{tr}_B\rho_{ABB'} = \psi_A \otimes \omega_{B'} \neq \psi_{AB}.$$

Extendibility hierarchy

Thus, extendibility can be understood as an obstruction to entanglement.

In fact, extendibility creates a *hierarchy* states (exercise):

Extendibility hierarchy

Every k-extendible state is also k'-extendible for $k' \leq k$.

Denoting by $\mathrm{Ext}_k(A:B)$ the set of k-extendible states on AB, we thus have

$$\mathrm{Ext}_1(A:B) \supset \mathrm{Ext}_2(A:B) \supset \mathrm{Ext}_3(A:B) \supset \cdots \supset \mathrm{Ext}_\infty(A:B) = \mathrm{Sep}(A:B).$$

The inclusion $\mathrm{Ext}_\infty(A:B) \supset \mathrm{Sep}(A:B)$ is the content of the following statement:

Proposition

Separable states are ∞-extendible.

Proof.

Let $\sigma_{AB} = \sum_i p_i \sigma_A^{(i)} \otimes \sigma_B^{(i)}$ be separable, then $\sigma_{AB_1 \cdots B_k} = \sum_i p_i \sigma_A^{(i)} \otimes \sigma_{B_1}^{(i)} \otimes \cdots \otimes \sigma_{B_k}^{(i)}$ defines a k-extension for arbitrary $k \in \mathbb{N}$. ☐

Entangled extendible states

Converse statement: Every ∞-extendible state is separable (see, e.g., [DPS04]).

This implies that for every entangled state ρ_{AB} there exists a k_0 such that ρ_{AB} has no k-extension for $k \geq k_0$.

An example of an entangled 2-extendible state is the following two-qubit isotropic state:

$$\rho_{AB}(1/2) = \frac{1}{2}\Phi^+_{AB} + \frac{1}{2}\frac{1}{4}\mathbb{1}_{AB}.$$

Since $x = 1/2 < 2/3$, this state is entangled by our results on isotropic states.

The following state is a 2-extension of the isotropic state ρ_{AB}:

$$\rho_{ABB'} = \frac{1}{4}\Phi^+_{AB} \otimes \mathbb{1}_{B'} + \frac{1}{4}\Phi^+_{AB'} \otimes \mathbb{1}_{B}.$$

We compute:

$$\mathrm{tr}_{B'}\rho_{ABB'} = \frac{1}{4}\Phi^+_{AB}\mathrm{tr}(\mathbb{1}_{B'}) + \frac{1}{4}\mathrm{tr}_{B'}(\Phi^+_{AB'}) \otimes \mathbb{1}_{B'} = \frac{1}{2}\Phi^+_{AB} + \frac{1}{4}\frac{1}{2}\mathbb{1}_A \otimes \mathbb{1}_B.$$

Extendibility of Werner and isotropic states

The extendibility of d-dimensional Werner and isotropic states was determined analytically in [JV13] using their symmetries:

- The Werner state $\rho^W_{AB} = \frac{1}{d(d^2-1)}[(d-\alpha)\mathbb{1} + (d\alpha - 1)\mathbb{F}]$ is k-extendible iff

$$\alpha \geq \frac{1-d}{k}. \tag{1}$$

  Recall that $\alpha \in [-1, 1]$. The k-extendibility condition (1) then implies that for $d \geq 3$ every Werner state is at least 2-extendible.

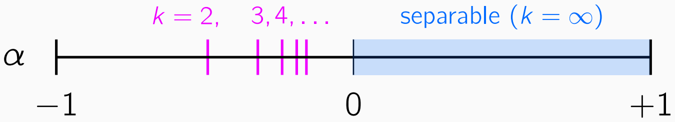
- The isotropic state $\rho^I_{AB} = (1-x)\Phi^+_{AB} + x\frac{1}{d^2}\mathbb{1}$ is k-extendible iff

$$x \geq \frac{d}{d+1}\left(1 - \frac{1}{k}\right). \tag{2}$$

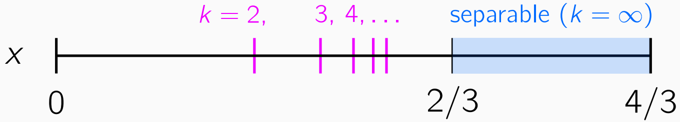
Recall that the Werner state ρ^W_{AB} is separable iff $\alpha \geq 0$, and the isotropic state ρ^I_{AB} is separable iff $x \geq \frac{d}{d+1}$. Both results also follow from taking the limit $k \to \infty$ in (1) and (2).

Extendibility of Werner and isotropic states for $d = 2$

Two-qubit Werner state $\rho_{AB}^W = \frac{1}{6}\left[(2-\alpha)\mathbb{1} + (2\alpha - 1)\mathbb{F}\right]$ is k-extendible iff $\alpha \geq -\frac{1}{k}$:



Two-qubit isotropic state $\rho_{AB}^I = (1-x)\Phi_{AB}^+ + x\frac{1}{4}\mathbb{1}$ is k-extendible iff $x \geq \frac{2}{3}\left(1 - \frac{1}{k}\right)$:



A de Finetti theorem for pure symmetric states

Measure on pure states

We will focus on pure states in the symmetric subspace

$$\begin{aligned}\mathrm{Sym}^n\left(\mathbb{C}^d\right) &= \left\{|\psi\rangle \in (\mathbb{C}^d)^{\otimes n} \,:\, \varphi(\pi)|\psi\rangle = |\psi\rangle\right\}\\ &= \mathrm{span}\left\{|\phi\rangle^{\otimes n} \,:\, |\phi\rangle \in \mathbb{C}^d\right\}.\end{aligned}$$

We have $\dim \mathrm{Sym}^n(\mathbb{C}^d) = \binom{n+d-1}{n}$ by Weyl's dimension formula.

Let $\Pi_{\mathrm{sym}} = \frac{1}{n!}\sum_{\pi\in S_n} \varphi(\pi)$ be the projector onto $\mathrm{Sym}^n\left(\mathbb{C}^d\right)$.

We will prove a different formula for Π_{sym} in terms of the following construction.

Normalized measure on pure states

Denote by $\mathcal{D}_1(\mathbb{C}^d)$ the set of pure states on $\mathbb{C}^d$. Parametrizing $|\phi\rangle = U|\phi_0\rangle$ for some fixed state $|\phi_0\rangle$ and unitary $U \in \mathcal{U}_d$, the Haar measure on $\mathcal{U}^d$ induces a normalized measure $\mathrm{d}\phi$ on $\mathcal{D}_1(\mathbb{C}^d)$ via

$$\int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi\, f(|\phi\rangle) = \int_{\mathcal{U}_d} \mathrm{d}U\, f(U|\phi_0\rangle).$$

Formula for symmetric subspace projector

Proposition

$$\Pi_{\mathrm{sym}} = \binom{n+d-1}{n} \int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi \, |\phi\rangle\langle\phi|^{\otimes n}.$$

Proof.

Let us first consider the operator $X = \int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi \, |\phi\rangle\langle\phi|^{\otimes n}$. The vectors $|\phi\rangle^{\otimes n}$ span the symmetric subspace $\mathrm{Sym}^n\left(\mathbb{C}^d\right)$, and thus X is fully supported on the symmetric subspace.

In addition, the operator X is invariant under the diagonal action $U^{\otimes n}$ for $U \in \mathcal{U}_d$:

$$\begin{aligned}
U^{\otimes n} X (U^\dagger)^{\otimes n} &= \int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi \, (U|\phi\rangle\langle\phi|U^\dagger)^{\otimes n} \\
&= \int_{\mathcal{U}_d} \mathrm{d}V \, (UV|\phi_0\rangle\langle\phi_0|V^\dagger U^\dagger)^{\otimes n} && \text{by definition of } \mathrm{d}\phi\,, \\
&= \int_{\mathcal{U}_d} \mathrm{d}V \, (V|\phi_0\rangle\langle\phi_0|V^\dagger)^{\otimes n} = X && \text{by invariance of the Haar measure.}
\end{aligned}$$

Formula for symmetric subspace projector

Proposition

$$\Pi_{\mathrm{sym}} = \binom{n+d-1}{n} \int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi \, |\phi\rangle\langle\phi|^{\otimes n}.$$

Proof continued.

Since $\mathrm{Sym}^n(\mathbb{C}^d)$ is irreducible under the representation $U^{\otimes n}$, it follows from Schur's Lemma that X is proportional to Π_{sym}, the identity on $\mathrm{Sym}^n(\mathbb{C}^d)$:

$$cX = \Pi_{\mathrm{sym}} \qquad \text{for some } c \geq 0.$$

Taking traces on both sides, we have $\mathrm{tr} X = \int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi \, \mathrm{tr} |\phi\rangle\langle\phi|^{\otimes n} = \int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi \; = 1$ by normalization of $\mathrm{d}\psi$ and thus

$$c = \mathrm{tr} \Pi_{\mathrm{sym}} = \dim \mathrm{Sym}^n(\mathbb{C}^d) = \binom{n+d-1}{n}.$$

☐

A de Finetti theorem for pure symmetric states

Recall the 1-norm $\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X}$, and induced trace distance on quantum states

$$D(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1.$$

We can now prove the following de Finetti theorem:

De Finetti theorem for pure symmetric states

Let $\mathcal{H}_{A_i} \cong \mathbb{C}^d$ and $|\psi\rangle_{A_1\cdots A_n} \in \mathrm{Sym}^n(\mathbb{C}^d)$. Then for any $k < n$,

$$D\left(\psi_{A_1\cdots A_k}, \int_{\mathcal{D}_1(\mathbb{C}^d)} \mathrm{d}\phi\, p_\psi(\phi) |\phi\rangle\langle\phi|^{\otimes k}\right) \leq \sqrt{\frac{dk}{n-k}},$$

where $p_\psi(\phi)$ is a probability density that depends on $|\psi\rangle$.

Proof of de Finetti theorem for pure symmetric states

Proof in lecture notes and given in lecture.

Extension to permutation-invariant mixed states

Symmetric purifications

New notation: If $\mathcal{H}_A \cong \mathbb{C}^d$ is a d-dim. quantum system, n copies of A have state space

$$\mathcal{H}_{A^n} = \mathcal{H}_{A_1 \dots A_n} = \mathcal{H}_A^{\otimes n}.$$

For a permutation $\pi \in S_n$ we then denote by $\pi_A = \varphi(\pi) \in \mathrm{End}(\mathcal{H}_{A^n})$ the corresponding permutation operator.

Recall that a state $\rho_{A_1 \cdots A_n}$ is called *permutation-invariant* if

$$\pi_A \rho_{A_1 \dots A_n} \pi_A^\dagger = \rho_{A_1 \dots A_n} \quad \text{for all } \pi \in S_n.$$

Goal

Generalize previous de Finetti theorem for pure symmetric states to arbitrary permutation-invariant states.

To prepare this generalization, we first relate permutation-invariant states to pure states in $\mathrm{Sym}^n(\mathbb{C}^d)$.

Symmetric purifications

Lemma

Let $\mathcal{H}_{A_i} = \mathbb{C}^d$ for $i = 1, \ldots, n$ and $\rho_{A_1 \cdots A_n}$ be permutation invariant. Then $\rho_{A_1 \cdots A_n}$ has a purification $|\psi^\rho\rangle \in \mathrm{Sym}^n(\mathbb{C}^d \otimes \mathbb{C}^d)$.

Proof in lecture notes and given in lecture.

De Finetti theorem for permutation-invariant states

We can now prove the following general de Finetti theorem:

De Finetti theorem

Let $\mathcal{H}_{A_i} = \mathbb{C}^d$ for $i = 1, \ldots, n$ and $\rho_{A_1 \cdots A_n}$ be permutation-invariant. For any $k < n$,

$$D\left(\rho_{A_1 \cdots A_k}, \int d\mu_\rho(\sigma)\, \sigma_A^{\otimes k}\right) \leq \sqrt{\frac{d^2 k}{n-k}},$$

where $d\mu_\rho(\sigma)$ is a measure on the space of mixed states on $\mathbb{C}^d$ that depends on ρ.

Proof.

Let $|\psi^\rho\rangle_{A^n R^n} \in \mathrm{Sym}^n(\mathbb{C}^d \otimes \mathbb{C}^d)$ be a symmetric purification of the permutation-invariant state ρ, which exists by the previous lemma.

Applying the pure-state de Finetti theorem to $|\psi^\rho\rangle$ gives the bound

$$D\left(\psi^\rho_{A_1 R_1 \cdots A_k R_k}, \int d\phi\, p_{\psi^\rho}(\phi) |\phi\rangle\langle\phi|_{AR}^{\otimes k}\right) \leq \sqrt{\frac{d^2 k}{n-k}}$$

with a suitable probability density $p_{\psi^\rho}(\varphi)$.

De Finetti theorem for permutation-invariant states

De Finetti theorem

Let $\mathcal{H}_{A_i} = \mathbb{C}^d$ for $i = 1, \dots, n$ and $\rho_{A_1 \cdots A_n}$ be permutation-invariant. For any $k < n$,

$$D\left(\rho_{A_1 \cdots A_k}, \int \mathrm{d}\mu_\rho(\sigma)\, \sigma_A^{\otimes k}\right) \leq \sqrt{\frac{d^2 k}{n-k}},$$

where $\mathrm{d}\mu_\rho(\sigma)$ is a measure on the space of mixed states on $\mathbb{C}^d$ that depends on ρ.

Proof.

The claim now follows from the monotonicity of $D(\cdot, \cdot)$ under the partial trace tr_{R^k}:

$$D\left(\rho_{A_1 \cdots A_k}, \int \mathrm{d}\phi\, p_{\psi^\rho}(\phi) \mathrm{tr}_{R^k} \phi_{AR}^{\otimes k}\right) \leq D\left(\psi^\rho_{A_1 R_1 \cdots A_k R_k}, \int \mathrm{d}\phi\, p_{\psi^\rho}(\phi) |\phi\rangle\langle\phi|_{AR}^{\otimes k}\right) \leq \sqrt{\frac{d^2 k}{n-k}},$$

which concludes the proof. □

References

[DPS04] Andrew C. Doherty, Pablo A. Parrilo, and Federico M. Spedalieri. **“Complete family of separability criteria”.** *Physical Review A* 69 (2 Feb. 2004), p. 022308. arXiv: `quant-ph/0308032`.

[JV13] Peter D. Johnson and Lorenza Viola. **“Compatible quantum correlations: Extension problems for Werner and isotropic states”.** *Physical Review A* 88 (3 Sept. 2013), p. 032323. arXiv: `1305.1342 [quant-ph]`.