UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Math 595 Representation-theoretic methods in QIT

Families of invariant states

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Werner states

Definition

Let $\mathcal{H}_A = \mathcal{H}_B \cong \mathbb{C}^d$ be d-dimensional Hilbert spaces $d \geq 2$. A quantum state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called a *Werner state* [Wer89] if

$$(U \otimes U)\rho_{AB}(U \otimes U)^\dagger = \rho_{AB} \qquad \text{for all } U \in \mathcal{U}_d.$$

Schur-Weyl decomposition for $n = 2$:

$$\left(\mathbb{C}^d\right)^{\otimes 2} = V_{\square\square} \otimes U^d_{\square\square} \oplus V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \otimes U^d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}.$$

Dimensions of the S_n-irreps $V_{\square\square}$ and $V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$:

$$d_{\square\square} = \frac{2!}{2 \cdot 1} = 1 \qquad\qquad d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{2!}{2 \cdot 1} = 1.$$

$$\implies \left(\mathbb{C}^d\right)^{\otimes 2} = U^d_{\square\square} \oplus U^d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}.$$

Schur-Weyl duality for $n = 2$

The representation space $U^d_{\ydiagram{2}}$ is equal to the **symmetric subspace**

$$\mathrm{Sym}^2(\mathbb{C}^d) = \left\{ |v\rangle \in (\mathbb{C}^d)^{\otimes 2} \,:\, \mathbb{F}|v\rangle = |v\rangle \right\}$$

$$m_{\ydiagram{2},d} = \dim \mathrm{Sym}^2(\mathbb{C}^d) = \frac{d(d+1)}{2}.$$

On the other hand, the representation space $U^d_{\ydiagram{1,1}}$ is equal to the **antisymmetric subspace**

$$\Lambda^2(\mathbb{C}^d) = \left\{ |v\rangle \in (\mathbb{C}^d)^{\otimes 2} \,:\, \mathbb{F}|v\rangle = -|v\rangle \right\}$$

$$m_{\ydiagram{1,1},d} = \dim \Lambda^2(\mathbb{C}^d) = \frac{d(d-1)}{2}.$$

Schur's lemma and $(U \otimes U)$-symmetry:

$$\rho_{AB} \cong c_{\ydiagram{2}} \mathbb{1}_{U^d_{\ydiagram{2}}} \oplus c_{\ydiagram{1,1}} \mathbb{1}_{U^d_{\ydiagram{1,1}}}$$

for some $c_{\ydiagram{2}}, c_{\ydiagram{1,1}} \geq 0$ with

$$\mathrm{tr}\rho = 1 = c_{\ydiagram{2}} \frac{d(d+1)}{2} + c_{\ydiagram{1,1}} \frac{d(d-1)}{2}.$$

Structure of Werner states

Young symmetrizers for $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, with standard Young tableaux $\boxed{1|2}$ and $\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}$:

$$e_{\boxed{1|2}} = \mathbb{1} + \mathbb{F} \qquad\qquad e_{\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix}} = \mathbb{1} - \mathbb{F}.$$

Projections:

$$P_{\square\square} = \frac{1}{2}(\mathbb{1} + \mathbb{F}) \qquad \text{onto } V_{\square\square} \otimes U^d_{\square\square} \equiv U^d_{\square\square}$$

$$P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{1}{2}(\mathbb{1} - \mathbb{F}) \qquad \text{onto } V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \otimes U^d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \equiv U^d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

with $\mathrm{tr} P_{\square\square} = \frac{d(d+1)}{2}$ and $\mathrm{tr} P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{d(d-1)}{2}$.

We have thus proved the following structure result for Werner states:

Werner states

A Werner state has the form

$$\rho_{AB} = x \frac{2}{d(d+1)} P_{\square\square} + (1-x) \frac{2}{d(d-1)} P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \qquad \text{for some } x \in [0,1].$$

Structure of Werner states

There is an alternative parametrization of a Werner state using the *visibility* $\alpha \coloneqq \mathrm{tr}(\rho_{AB}\mathbb{F})$:

$$\rho_{AB} = \frac{1}{d(d^2-1)}\left[(d-\alpha)\mathbb{1} + (d\alpha - 1)\mathbb{F}\right].$$

Recall the twirling operation

$$\mathcal{T}(X) = \int_{\mathcal{U}_d} \mathrm{d}U\,(U\otimes U)X(U\otimes U)^\dagger,$$

where $\mathrm{d}U$ denotes the Haar measure on $\mathcal{U}_d$.

Properties of Werner states

(i) Every Werner state is invariant under $\mathcal{T}$.

(ii) Let ρ_{AB} be an arbitrary state. Then $\mathcal{T}(\rho_{AB})$ is a Werner state of visibility $\alpha = \mathrm{tr}(\mathbb{F}\rho_{AB})$.

Entanglement in Werner states

A Werner state ρ_{AB} is entangled iff $\mathrm{tr}(\rho_{AB}\mathbb{F}) < 0$.

Proof just as in the qubit case (see lecture notes for details).

Multipartite Werner states

Multipartite Werner states

Definition

Let $\mathcal{H}_{A_i} = \mathbb{C}^d$ for $i = 1, \ldots, n$. A state $\rho_{A_1 \cdots A_n}$ is called a *multipartite Werner state* if

$$U_A^{\otimes n} \rho_{A_1 \cdots A_n} (U_A^\dagger)^{\otimes n} = \rho_{A_1 \cdots A_n} \qquad \text{for all } U_A \in \mathcal{U}_d.$$

Schur-Weyl duality: With $Q_\pi := \varphi(\pi) \in \mathrm{End}(A^n)$,

$$\rho_{A_1 \cdots A_n} = \sum_{\pi \in S_n} c_\pi Q_\pi \qquad \text{for some } c_\pi \in \mathbb{C}.$$

Special case $n = 2$ (bipartite Werner states): $\rho_{A_1 A_2} = \alpha \mathbb{1} + \beta \mathbb{F}$.

However, these expressions for $\rho_{A_1 \cdots A_n}$ may not always be useful since the Q_π are in general not positive semi-definite.

Structure of multipartite Werner states

Schur-Weyl decomposition: $(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash_d n} V_\lambda \otimes U_\lambda^d$.

Schur's lemma and $U^{\otimes n}$-invariance:

$$\rho_{A_1 \cdots A_n} = \bigoplus_{\lambda \vdash_d n} x_\lambda \rho_\lambda \otimes \frac{1}{m_{\lambda,d}} \mathbb{1}_{U_\lambda^d}$$

with prob. dist. $(x_\lambda)_{\lambda \vdash_d n}$, quantum states ρ_λ on V_λ for $\lambda \vdash_d n$, and $m_{\lambda,d} = \dim U_\lambda^d$.

If in addition $\rho_{A_1 \cdots A_n}$ is permutation-invariant, $Q_\pi \rho_{A_1 \cdots A_n} Q_\pi^\dagger = \rho_{A_1 \cdots A_n}$ for all $\pi \in S_n$, then

$$\rho_{A_1 \cdots A_n} = \bigoplus_{\lambda \vdash_d n} x_\lambda \frac{1}{d_\lambda} \mathbb{1}_{V_\lambda} \otimes \frac{1}{m_{\lambda,d}} \mathbb{1}_{U_\lambda^d} = \sum_{\lambda \vdash_d n} x_\lambda \tau_\lambda,$$

where $\tau_\lambda = \frac{1}{d_\lambda m_{\lambda,d}} \Pi_\lambda$ and Π_λ is the isotypical projector onto $V_\lambda \otimes U_\lambda^d$.

Isotropic states

Isotropic states

Definition

A state ρ_{AB} on systems AB with $\mathcal{H}_A \cong \mathcal{H}_B \cong \mathbb{C}^d$ is called *isotropic* [HH99] if

$$\left(U \otimes \overline{U}\right) \rho_{AB} \left(U \otimes \overline{U}\right)^\dagger = \rho_{AB} \quad \text{for all } U \in \mathcal{U}_d.$$

Isotropic states are important, since they are the **Choi operators** (see, e.g., [Led23]) of depolarizing channels $\mathcal{D}(X) = (1-q)X + q\,\mathrm{tr}(X)\frac{1}{d}\mathbb{1}_d$ satisfying

$$\mathcal{D}(UXU^\dagger) = U\mathcal{D}(X)U^\dagger \qquad \text{for all } U \in \mathcal{U}_d.$$

What operators are invariant under $(U \otimes \overline{U})$?

(i) Identity $\mathbb{1}_{AB}$ (trivially).

(ii) Maximally entangled state: $\left(U \otimes \overline{U}\right)|\Phi^+\rangle_{AB} = |\Phi^+\rangle_{AB} \qquad \text{for all } U \in \mathcal{U}_d.$

(ii) follows from the "transpose trick" $(X \otimes \mathbb{1})|\Phi^+\rangle_{AB} = (\mathbb{1} \otimes X^T)|\Phi^+\rangle_{AB}$ (exercise).

Structure of isotropic states

Observe that a state ρ_{AB} is isotropic iff $\rho_{AB}^{T_B}$ is a Werner state:

$$(U \otimes U)\rho_{AB}^{T_B}(U^\dagger \otimes U^\dagger) = \left[\left(U \otimes \overline{U}\right)\rho_{AB}\left(U \otimes \overline{U}\right)^\dagger\right]^{T_B} = \rho_{AB}^{T_B}.$$

where the first equality follows from the general identity

$$\left[(X_1 \otimes Y_1)\, Z_{AB}\, (X_2 \otimes Y_2)\right]^{T_B} = (X_1 \otimes Y_2^T)\, Z_{AB}^{T_B}\, (X_2 \otimes Y_1^T).$$

Expanding $\rho_{AB}^{T_B} = \alpha \mathbb{1}_{AB} + \beta \mathbb{F}_{AB}$, and using $(\Phi_{AB}^+)^{T_B} = \frac{1}{d}\mathbb{F}_{AB}$, we have the following result:

Structure of isotropic states

$$\rho_{AB} = (1 - x)|\Phi^+\rangle\langle\Phi^+|_{AB} + x\frac{1}{d^2}\mathbb{1}_{AB} \qquad \text{for } x \in \left[0, \frac{d^2}{d^2 - 1}\right].$$

ρ_{AB} is PSD iff $x \in \left[0, \frac{d^2}{d^2-1}\right]$.

Entanglement in isotropic states

We can consider the following twirling operation:

$$\mathcal{T}(X) = \int_{\mathcal{U}_d} \mathrm{d}U \left(U \otimes \overline{U}\right) X \left(U \otimes \overline{U}\right)^\dagger$$

Entanglement in isotropic states

Let $\rho_{AB}(x) := (1 - x)\Phi^+_{AB} + x\frac{1}{d^2}\mathbb{1}_{AB}$ with $x \in \left[0, \frac{d^2}{d^2-1}\right]$ be an isotropic state.

(i) Let σ_{AB} be arbitrary with $\beta := \mathrm{tr}(\sigma_{AB}\Phi^+_{AB}) = \langle\Phi^+|\sigma_{AB}|\Phi^+\rangle$. Then

$$\int_{\mathcal{U}_d} (U \otimes \overline{U})\sigma_{AB}(U \otimes \overline{U})^\dagger = \rho_{AB}(y),$$

where $y = \frac{d^2}{d^2-1}(1 - \beta)$.

(ii) ρ_{AB} is separable iff $x \geq \frac{d}{d+1}$.

Entanglement in isotropic states

Entanglement in isotropic states

Comparing Werner and isotropic states

For 2 qubits, Werner and isotropic states have an equivalent entanglement structure (see exercises):

Two-qubit Werner and isotropic states

Any Werner state on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is local unitary equivalent to an isotropic state.

However, for local dimension $d \geq 3$ the two families of states are not even equivalent via a global unitary except for the completely mixed state $\frac{1}{d^2}\mathbb{1}_{AB}$, which obviously has both $(U \otimes U)$ and $(U \otimes \overline{U})$-symmetry.

References

[HH99] Michał Horodecki and Paweł Horodecki. **“Reduction criterion of separability and limits for a class of distillation protocols”.** *Physical Review A* 59.6 (June 1999), pp. 4206–4216. quant-ph: quant-ph/9708015.

[Led23] Felix Leditzky. **“Quantum channels”.** Lecture notes. 2023. eprint: https://www.overleaf.com/project/6052c89e7f1a335d1d49d099.

[Wer89] Reinhard F. Werner. **“Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model”.** *Physical Review A* 40 (8 1989), pp. 4277–4281.