



UNIVERSITY OF
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Math 595 Representation-theoretic methods in QIT

Schur-Weyl duality

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Representations of direct product groups

Direct products of groups

Definition

The *direct product* $G \times H$ of two groups G and H is a group with underlying set $G \times H$ and multiplication defined for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$ as

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Definition

Let (φ, V) and (ψ, W) be representations of groups G and H respectively. Then $V \hat{\otimes} W$ affords the *external product representation* of the direct product $G \times H$ by defining

$$(\varphi \hat{\otimes} \psi)(g, h) := \varphi(g) \otimes \psi(h).$$

Facts:

- (i) If (φ, V) and (ψ, W) are irreducible, then so is $(\varphi \hat{\otimes} \psi, V \hat{\otimes} W)$ (exercise).
- (ii) Every irreducible representation of $G \times H$ arises this way.

Commutants of endomorphism algebras

The commutant of an algebra

In this section we follow the exposition in [Chr06].

Definition

Let S be a subset of an algebra \mathcal{A} . The commutant S' of S is the collection of those elements in \mathcal{A} commuting with all of S :

$$S' = \{a \in \mathcal{A} : as = sa \text{ for all } s \in S\}.$$

For a vector space V , the set of operators $\text{End}(V)$ acting on V is an algebra with respect to addition, scalar multiplication, and composition of operators.

A **representation** (φ, V) of an algebra \mathcal{A} is an algebra homomorphism $\varphi : \mathcal{A} \rightarrow \text{End}(V)$.

Lemma

Let V and W be finite-dimensional complex vector spaces. The commutant of $\text{End}(V) \otimes \mathbb{1}_W$ in $\text{End}(V \otimes W) \cong \text{End}(V) \otimes \text{End}(W)$ is $\mathbb{1}_V \otimes \text{End}(W)$.

Proof.

Set $\mathcal{A} = \text{End}(V) \otimes \mathbb{1}_W$ and $\mathcal{B} = \mathbb{1}_V \otimes \text{End}(W)$. Clearly, an element $\mathbb{1}_V \otimes b \in \mathcal{B}$ commutes with every element $a \otimes \mathbb{1}_W \in \mathcal{A}$, and hence $\mathcal{B} \subset \mathcal{A}'$.

Suppose now that $a \otimes \mathbb{1}_W \in \mathcal{A}$ and $b \in \mathcal{A}'$ are arbitrary. Let $\dim W = n$ and write

$$a \otimes \mathbb{1}_W = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & a \end{pmatrix} \qquad b = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

with $b_{ij} \in \text{End}(V)$.

Lemma

Let V and W be finite-dimensional complex vector spaces. The commutant of $\text{End}(V) \otimes \mathbb{1}_W$ in $\text{End}(V \otimes W) \cong \text{End}(V) \otimes \text{End}(W)$ is $\mathbb{1}_V \otimes \text{End}(W)$.

Proof.

Then $(a \otimes \mathbb{1}_W)b = b(a \otimes \mathbb{1}_W)$ is equivalent to

$$\begin{pmatrix} ab_{11} & ab_{12} & \cdots & ab_{1n} \\ ab_{21} & ab_{22} & \cdots & ab_{2n} \\ \vdots & & \ddots & \vdots \\ ab_{n1} & ab_{n2} & \cdots & ab_{nn} \end{pmatrix} = \begin{pmatrix} b_{11}a & b_{12}a & \cdots & b_{1n}a \\ b_{21}a & b_{22}a & \cdots & b_{2n}a \\ \vdots & & \ddots & \vdots \\ b_{n1}a & b_{n2}a & \cdots & b_{nn}a \end{pmatrix}$$

Hence, for fixed i, j , we have $[a, b_{ij}] = 0$ for all $a \in \text{End}(V)$. This forces b_{ij} to be a multiple of the identity, $b_{ij} = \lambda_{ij} \mathbb{1}_A$ for some $\lambda_{ij} \in \mathbb{C}$. Let $\tilde{b} \in \text{End}(W)$ be defined by $(\tilde{b})_{ij} = \lambda_{ij}$, then $b = \mathbb{1}_V \otimes \tilde{b} \in \mathbb{1}_V \otimes \text{End}(W) = \mathcal{B}$, and thus $\mathcal{A}' \subset \mathcal{B}$. \square

Duality theorem

Duality theorem

Let (φ, V) be a representation of a finite group G with isotypical decomposition

$$V = \bigoplus_{\alpha} V_{\alpha} \otimes \mathbb{C}^{n_{\alpha}}$$

into pairwise inequivalent irreducible representations $(\varphi_{\alpha}, V_{\alpha})$ with multiplicity n_{α} . Let $\mathcal{A} \subset \text{End}(V)$ be the subalgebra generated by φ , and set $\mathcal{B} = \mathcal{A}'$. Then:

- (i) $\mathcal{A} \cong \bigoplus_{\alpha} \text{End}(V_{\alpha}) \otimes \mathbb{1}_{\mathbb{C}^{n_{\alpha}}}$
- (ii) $\mathcal{B} \cong \bigoplus_{\alpha} \mathbb{1}_{V_{\alpha}} \otimes \text{End}(\mathbb{C}^{n_{\alpha}})$
- (iii) $\mathcal{B}' = (\mathcal{A}')' = \mathcal{A}$

Needed for the proof: Set $d_{\alpha} = \dim V_{\alpha}$, pick unitary matrix representations for the irreps $(\varphi_{\alpha}, V_{\alpha})$, and denote by $\varphi_{\alpha,ij}$ their matrix coefficients. By Schur's Lemma,

$$\frac{1}{|G|} \sum_{g \in G} \overline{\varphi_{\alpha,ij}(g)} \varphi_{\beta,kl}(g) = \frac{1}{d_{\alpha}} \delta_{\alpha\beta} \delta_{ik} \delta_{jl}.$$

Proof of the Duality theorem

(i) $\mathcal{A} \cong \bigoplus_{\alpha} \text{End}(V_{\alpha}) \otimes \mathbb{1}_{\mathbb{C}^{n_{\alpha}}}$:

Set $d_{\alpha} = \dim V_{\alpha}$. Choosing the φ_{α} unitary and using the preceding result on orthogonality of matrix coefficients,

$$\mathcal{A} \ni \frac{d_{\alpha}}{|G|} \sum_{g \in G} \overline{\varphi_{\alpha,ij}(g)} \varphi(g) = E_{ij}^{(\alpha)} \otimes \mathbb{1}_{\mathbb{C}^{n_{\alpha}}},$$

where $E_{ij}^{(\alpha)}$ is the (i, j) -elementary matrix in $\text{End}(V_{\alpha})$. Since the $E_{ij}^{(\alpha)}$ are a basis of $\text{End}(V_{\alpha})$,

$$\bigoplus_{\alpha} \text{End}(V_{\alpha}) \otimes \mathbb{1}_{\mathbb{C}^{n_{\alpha}}} \subset \mathcal{A}.$$

The reverse inclusion follows from the decomposition of V into isotypical components $V_{\alpha} \otimes \mathbb{C}^{n_{\alpha}}$, and hence we have equality.

Proof of the Duality theorem

(ii) $\mathcal{B} \cong \bigoplus_{\alpha} \mathbb{1}_{V_{\alpha}} \otimes \text{End}(\mathbb{C}^{n_{\alpha}})$:

First we show that $\mathcal{B} \subset \bigoplus_{\alpha} \mathbb{1}_{V_{\alpha}} \otimes \text{End}(\mathbb{C}^{n_{\alpha}})$. To this end, let P_{α} be the projection onto V_{α} , that is, $P_{\alpha}\mathcal{A} = V_{\alpha} \otimes \mathbb{C}^{n_{\alpha}}$. Note that we have $\sum_{\alpha} P_{\alpha} = \mathbb{1}_{\mathcal{A}}$. Since every $b \in \mathcal{B}$ commutes with P_{α} by definition,

$$b = \mathbb{1}_{\mathcal{A}}b = \sum_{\alpha} P_{\alpha}b = \sum_{\alpha} P_{\alpha}bP_{\alpha} = \sum_{\alpha} b_{\alpha},$$

where $b_{\alpha} \in \text{End}(V_{\alpha} \otimes \mathbb{C}^{n_{\alpha}})$. By the preceding lemma, $b_{\alpha} = \mathbb{1}_{V_{\alpha}} \otimes b'_{\alpha}$ for some $b'_{\alpha} \in \text{End}(\mathbb{C}^{n_{\alpha}})$, and hence $\mathcal{B} \subset \bigoplus_{\alpha} \mathbb{1}_{V_{\alpha}} \otimes \text{End}(\mathbb{C}^{n_{\alpha}})$. The other inclusion holds since any $\bigoplus_{\alpha} \mathbb{1}_{V_{\alpha}} \otimes b_{\alpha}$ with $b_{\alpha} \in \text{End}(\mathbb{C}^{n_{\alpha}})$ commutes with $\bigoplus_{\alpha} a_{\alpha} \otimes \mathbb{1}_{\mathbb{C}^{n_{\alpha}}} \in \mathcal{A}$ by construction.

(iii) This follows by a similar argument as that in (ii) (exercise).

The Schur-Weyl decomposition

Tensor representations of the symmetric and unitary groups

We now focus on the following two groups:

- the symmetric group S_n , the set of bijections from $\{1, \dots, n\}$ to itself.
- the unitary group $\mathcal{U}_d = \{U \in \mathcal{L}(\mathbb{C}^d) : U^\dagger U = U U^\dagger = \mathbb{1}_d\}$.

The symmetric group has a representation on $(\mathbb{C}^d)^{\otimes n}$ by permuting tensor factors:

$$\varphi(\pi)(|\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle) = |\psi_{\pi^{-1}(1)}\rangle \otimes \dots \otimes |\psi_{\pi^{-1}(n)}\rangle.$$

The unitary group also has a representation on $(\mathbb{C}^d)^{\otimes n}$ by acting diagonally:

$$\omega(U)(|\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle) = U|\psi_1\rangle \otimes \dots \otimes U|\psi_n\rangle.$$

Definition

The *symmetric subspace* $\text{Sym}^n(V)$, also called the *n-th symmetric power* of V , is the subspace invariant under the action of S_n :

$$\text{Sym}^n(V) = (V^{\otimes n})^{S_n} = \{ |v\rangle \in V^{\otimes n} : \varphi(\pi)|v\rangle = |v\rangle \text{ for all } \pi \in S_n \}.$$

With $P = \frac{1}{n!} \sum_{\pi \in S_n} \varphi(\pi)$, we have $\text{Sym}^n(V) = PV^{\otimes n}$.

Generating set for the symmetric subspace

$$\text{Sym}^n(V) = \text{span}\{|v\rangle^{\otimes n} : |v\rangle \in V\}.$$

Proof of the proposition

The inclusion $\text{span}\{|v\rangle^{\otimes n} : |v\rangle \in V\} \subset \text{Sym}^n(V)$ follows from the observation that $\varphi(\pi)|v\rangle^{\otimes n} = |v\rangle^{\otimes n}$ for all $v \in V$ and $\pi \in S_n$.

For the other inclusion, let $\{|e_i\rangle\}_{i=1}^d$ be an orthonormal basis for V , with $d = \dim V$. By definition $\text{Sym}^n(V)$ is spanned by the vectors

$$|v_{i_1 \dots i_n}\rangle := \sum_{\pi \in S_n} \varphi(\pi)(|e_{i_1}\rangle \otimes \dots \otimes |e_{i_n}\rangle) = \sum_{\pi \in S_n} |e_{i_{\pi^{-1}(1)}}\rangle \otimes \dots \otimes |e_{i_{\pi^{-1}(n)}}\rangle$$

for indices $i_j \in \{1, \dots, d\}$ $j = 1, \dots, n$.

Now rewrite the vectors $|v_{i_1 \dots i_n}\rangle$ using partial derivatives as (see exercises).

$$|v_{i_1 \dots i_n}\rangle = \partial_{\lambda_2} \dots \partial_{\lambda_n} \left(|e_{i_1}\rangle + \sum_{j=2}^n \lambda_j |e_{i_j}\rangle \right)^{\otimes n} \Big|_{\lambda_2 = \dots = \lambda_n = 0}.$$

Proof of the proposition

This calculation involves a sequence of partial derivatives of the form

$$\left. \partial_{\lambda_j} (|v\rangle + \lambda_j |e_j\rangle)^{\otimes n} \right|_{\lambda_j=0} = \lim_{\lambda_j \rightarrow 0} \frac{(|v\rangle + \lambda_j |e_j\rangle)^{\otimes n} - |v\rangle^{\otimes n}}{\lambda_j}.$$

The $|v_{i_1 \dots i_n}\rangle$ are thus limits of elements in $W = \text{span}\{|v\rangle^{\otimes n} : |v\rangle \in V\}$. Since W is finite-dimensional and closed in $\text{Sym}^n(V)$, we have $|v_{i_1 \dots i_n}\rangle \in W$ for all indices i_1, \dots, i_n , and thus $\text{Sym}^n(V) \subset W$.

Symmetric subspace of operators

Symmetric subspace of operators

Let $C \in \text{End}(V^{\otimes n})$ be such that $\varphi(\pi)C\varphi(\pi)^\dagger = C$ for all $\pi \in S_n$. Then

$$C \in \text{span}\{X^{\otimes n} : X \in \text{End}(V)\}.$$

Proof.

Let $W = \text{End}(V^{\otimes n}) \cong \text{End}(V)^{\otimes n}$ and let $\{|e_i\rangle\}_{i=1}^d$ be a fixed basis of V . Consider the basis $\{E_{ij}\}_{i,j=1}^d$ of $\text{End}(V)$, where $E_{ij} : |e_k\rangle \mapsto \delta_{jk} |e_i\rangle$. Denote by $\varphi : S_n \rightarrow \text{GL}(V^{\otimes n})$ the tensor representation of S_n on $V^{\otimes n}$ and by $\tilde{\varphi} : S_n \rightarrow \text{GL}(W)$ the analogous tensor representation of S_n on $W = \text{End}(V)^{\otimes n}$. Then $\tilde{\varphi}(\pi)$ acting on $X \in \text{End}(V^{\otimes n})$ has the matrix representation $\varphi(\pi)X\varphi(\pi)^{-1}$. The claim then follows from the preceding lemma applied to $(\tilde{\varphi}, W)$. \square

The commutant theorem

In what follows we view $\omega : X \mapsto X^{\otimes n}$ as a representation of $\text{GL}(V)$.

Representations of the unitary and general linear group

A representation of $\mathcal{U}(V)$ is irreducible if and only if the corresponding representation of $\text{GL}(V)$ is irreducible.

This holds since $\text{GL}(V)$ is the “complexification” of $U(V)$ (more details: [CSM95; Pro07]).

Commutants of S_n and $\text{GL}(V)$

S_n and $\text{GL}(V)$ span each other's commutants in $\text{End}(V^{\otimes n})$.

Before proving this result, let's ponder what it means:

- Every operator $X \in \text{End}(V^{\otimes n})$ commuting with all $M^{\otimes n}$ ($M \in \text{End}(V)$) can be written as

$$X = \sum_{\pi \in S_n} c_{\pi} \varphi(\pi) \quad \text{for suitable } c_{\pi} \in \mathbb{C}.$$

- Every permutation-invariant operator $Y \in \text{End}(V^{\otimes n})$ can be written as

$$Y = \sum_i a_i X_i^{\otimes n} \quad \text{for suitable } a_i \in \mathbb{C} \text{ and } X_i \in \text{End}(V).$$

Proof of the commutant theorem

Commutants of S_n and $GL(V)$

S_n and $GL(V)$ span each other's commutants in $\text{End}(V^{\otimes n})$.

Proof.

Consider the following subalgebras of $\text{End}(V^{\otimes n})$:

$$\mathcal{A} := \text{span}\{\varphi(\pi) : \pi \in S_n\} \subset \text{End}(V^{\otimes n})$$

$$\mathcal{B} := \text{span}\{\omega(g) : g \in GL(V)\} \subset \text{End}(V^{\otimes n}).$$

Since $\varphi(\pi)$ and $\omega(U)$ commute for all $\pi \in S_n$, $U \in \mathcal{U}_d$, we have $\mathcal{B} \subset \mathcal{A}'$. The previous corollary shows that $\mathcal{A}' = \text{span}\{X^{\otimes n} : X \in \text{End}(V)\}$. Let $X \in \text{End}(V)$, then $X + t\mathbb{1}$ is invertible for all but finitely many t , and so $(X + t\mathbb{1})^{\otimes n} \in \mathcal{B}$ for all but finitely many t .

But $(X + t\mathbb{1})^{\otimes n}$ is a polynomial in t of degree n , and by Lagrange's interpolation theorem determined by any $n + 1$ distinct points. Hence, $(X + t\mathbb{1})^{\otimes n} \in \mathcal{B}$ for all t , in particular for $t = 0$. It follows that $\mathcal{A}' = \text{span}\{X^{\otimes n} : X \in \text{End}(V)\} \subset \mathcal{B}$, hence $\mathcal{A}' = \mathcal{B}$. The Double Commutant theorem now implies $\mathcal{B}' = \mathcal{A}$, which concludes the proof. \square

Schur-Weyl duality

Let $V = \mathbb{C}^d$ and $(\varphi, V^{\otimes n})$ and $(\omega, V^{\otimes n})$ be the tensor representations of S_n and $GL(V)$ defined before. As a representation of $S_n \times GL(V)$, the space $V^{\otimes n}$ decomposes as

$$V^{\otimes n} = \bigoplus_{\lambda} V_{\lambda} \otimes U_{\lambda},$$

where $(\varphi_{\lambda}, V_{\lambda})$ and $(\omega_{\lambda}, U_{\lambda})$ are inequivalent irreducible representations of S_n and $GL(V)$, respectively, and

$$\begin{aligned}\varphi(\pi) &= \bigoplus_{\lambda} \varphi_{\lambda}(\pi) \otimes \mathbb{1}_{U_{\lambda}} && \text{for } \pi \in S_n \\ \omega(g) &= \bigoplus_{\lambda} \mathbb{1}_{V_{\lambda}} \otimes \omega_{\lambda}(g) && \text{for } g \in GL(V).\end{aligned}$$

The same assertion holds when $GL(V)$ is replaced with $\mathcal{U}(V)$.

This follows from the “double commutant theorem” and the fact that S_n and $GL(V)$ span each other’s commutant. It remains to show that $U_{\lambda} \cong \text{Hom}_{S_n}(V_{\lambda}, V^{\otimes n})$ is an irreducible representation of $GL(V)$ (or \mathcal{U}_d), see lecture notes.

Schur-Weyl duality

$$V^{\otimes n} = \bigoplus_{\lambda} V_{\lambda} \otimes U_{\lambda}^d \quad \text{with } \varphi(\pi) = \bigoplus_{\lambda} \varphi_{\lambda}(\pi) \otimes \mathbb{1}_{U_{\lambda}^d} \quad \text{for } \pi \in S_n$$
$$\omega(g) = \bigoplus_{\lambda} \mathbb{1}_{V_{\lambda}} \otimes \omega_{\lambda}(g) \quad \text{for } g \in \text{GL}(V).$$

Denote by Π_{λ} the projection onto the isotypical component $V_{\lambda} \otimes U_{\lambda}^d$.

If $\rho \in \text{End}(V^{\otimes n})$ is permutation-invariant, $[\rho, \varphi(\pi)] = 0$ for all $\pi \in S_n$, then the operator

$$\rho_{\lambda, \lambda'} = \Pi_{\lambda} \rho \Pi_{\lambda'}$$

also commutes with all $\varphi(\pi)$ (since Π_{λ} does). Schur's Lemma implies the following:

- $\rho_{\lambda, \lambda'} = 0$ for $\lambda \neq \lambda'$.
- $\rho_{\lambda, \lambda} = \mathbb{1}_{V_{\lambda}} \otimes \rho_{\lambda}$ for some $\rho_{\lambda} \in \text{End}(U_{\lambda}^d)$.

Consequences of Schur-Weyl duality

Hence, a permutation-invariant state is block-diagonal with respect to the Schur-Weyl decomposition:

$$\rho \cong \bigoplus_{\lambda} \mathbb{1}_{V_{\lambda}} \otimes \rho_{\lambda} \quad \text{for some } \rho_{\lambda} \in \text{End}(U_{\lambda}^d).$$

By the same argument, a state commuting with all unitaries of the form $U^{\otimes n}$ with $U \in \mathcal{U}_d$ (a so-called **multipartite Werner state**) is of the form

$$\rho \cong \bigoplus_{\lambda} \tilde{\rho}_{\lambda} \otimes \mathbb{1}_{U_{\lambda}^d} \quad \text{for some } \tilde{\rho}_{\lambda} \in \text{End}(V_{\lambda}).$$

If a state has both symmetries, $[\rho, \varphi(\pi)] = 0$ for all $\pi \in S_n$ and $[\rho, U^{\otimes n}] = 0$ for all $U \in \mathcal{U}_d$, then both statements above apply, and we have

$$\rho \cong \bigoplus_{\lambda} c_{\lambda} \mathbb{1}_{V_{\lambda}} \otimes \mathbb{1}_{U_{\lambda}^d} \quad \text{for some } c_{\lambda} \in \mathbb{C}.$$

Since this is a block decomposition, ρ is positive semidefinite iff $c_{\lambda} \geq 0$ for all λ .

References

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