Math 595 Representation-theoretic methods in QIT

Basics from representation theory

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Motivation: Entanglement in Werner states

Symmetries reduce complexity

Previous section: There are efficient ways to detect entanglement (such as PPT criterion), but these are generally only *necessary* and not sufficient (*bound entanglement*).

In general, deciding whether a state is separably is NP-hard (believed to be impossible using efficient algorithms assuming P ≠ NP).

Central theme of this course: Symmetries reduce complexity of an object and make it easier to study its properties!

Example in this section: Entanglement in Werner states.

Werner states

Werner states are a class of symmetric quantum states, initially introduced to study hidden variable models [Wer89].

Werner state

A bipartite quantum state ρ_{AB} on $\mathbb{C}^d \otimes \mathbb{C}^d$ is called *Werner state* if

$$\rho_{AB} = (U \otimes U)\,\rho_{AB}\,(U \otimes U)^\dagger \quad \text{for all } U \in \mathcal{U}_d.$$

$\mathcal{U}_d = \{U \in \mathcal{L}(\mathbb{C}^d) : U^\dagger U = \mathbb{1}\}$ is the **group of unitaries on** $\mathbb{C}^d$.

Operational interpretation: Werner states are invariant under coordinated local basis transformations on each system.

In this section, we restrict our attention to two-qubit systems with $d = 2$.

Characterizing Werner states

Symmetry property of Werner states eliminates most degrees of freedom in the two-qubit state ρ_{AB}:

Two-qubit Werner states

Every quantum state ρ_{AB} on $\mathbb{C}^2 \otimes \mathbb{C}^2$ satisfying $\rho_{AB} = (U \otimes U)\,\rho_{AB}\,(U \otimes U)^\dagger$ for all $U \in \mathcal{U}_2$ can be written in terms of a single parameter $x \in [-1, 1]$ as

$$\rho_{AB} = \frac{2 - x}{6}\mathbb{1}_{AB} + \frac{2x - 1}{6}\mathbb{F}_{AB}.$$

Proof sketch.

Best proved using representation theory… (see later sections).

$\square$

Correlations in Werner states

In order to study correlations in Werner states, we want to have a means of *imposing* the $U \otimes U$-symmetry on an arbitrary (unsymmetric) state.

Idea: Average (or *twirl*) a state with respect to a uniform probability measure on $\mathcal{U}_d$, provided it exists!

Haar measure on $\mathcal{U}_d$

There exists a unique probability measure $\mathrm{d}U$ on $\mathcal{U}_d$ called *Haar measure* with the following properties:

(i) Normalization: $\int_{\mathcal{U}_d} \mathrm{d}U = 1$.

(ii) Left- and right-invariance: For any function f and an arbitrary unitary $V \in \mathcal{U}_d$,

$$\int_{\mathcal{U}_d} \mathrm{d}U\, f(VU) = \int_{\mathcal{U}_d} \mathrm{d}U\, f(UV) = \int_{\mathcal{U}_d} \mathrm{d}U\, f(U).$$

Werner twirl

Twirling operation: $\mathcal{T}(X_{AB}) = \int_{\mathcal{U}_2} \mathrm{d}U\,(U \otimes U)X_{AB}(U \otimes U)^\dagger$.

For any X_{AB} the twirled operator $\mathcal{T}(X_{AB})$ is a Werner state: For $V \in \mathcal{U}_2$,

$$\begin{aligned}
(V \otimes V)\mathcal{T}(X_{AB})(V \otimes V)^\dagger &= \int_{\mathcal{U}_2} \mathrm{d}U\,(V \otimes V)(U \otimes U)X_{AB}(U \otimes U)^\dagger(V \otimes V)^\dagger \\
&= \int_{\mathcal{U}_2} \mathrm{d}U\,(VU \otimes VU)X_{AB}(VU \otimes VU)^\dagger \\
&= \int_{\mathcal{U}_2} \mathrm{d}U\,(U \otimes U)X_{AB}(U \otimes U)^\dagger \\
&= \mathcal{T}(X_{AB}),
\end{aligned}$$

where we used linearity of the integral in the first equality, and the left-invariance of the Haar measure in the third equality.

Werner twirl

Let ρ_{AB} be a two-qubit density operator, then $\mathcal{T}(\rho_{AB})$ is a Werner state:

$$\mathcal{T}(\rho_{AB}) = \frac{2-x}{6}\mathbb{1}_{AB} + \frac{2x-1}{6}\mathbb{F}_{AB}.$$

Goal: Determine x.

First step: Compute overlap of $\mathcal{T}(\rho_{AB})$ with swap operator $\mathbb{F}_{AB}$.

$$\begin{aligned}
\operatorname{tr}\left[\mathbb{F}_{AB}\mathcal{T}(X_{AB})\right] &= \int_{\mathcal{U}_2} \mathrm{d}U \operatorname{tr}\left[\mathbb{F}_{AB}(U\otimes U)X_{AB}(U\otimes U)^\dagger\right] && \text{(linearity of trace)}\\
&= \int_{\mathcal{U}_2} \mathrm{d}U \operatorname{tr}\left[(U\otimes U)^\dagger\mathbb{F}_{AB}(U\otimes U)X_{AB}\right] && \text{(cyclicity of trace)}\\
&= \int_{\mathcal{U}_2} \mathrm{d}U \operatorname{tr}\left[\mathbb{F}_{AB}X_{AB}\right] && \text{(invariance of } \mathbb{F}\text{; see next slide)}\\
&= \operatorname{tr}\left[\mathbb{F}_{AB}X_{AB}\right]. && \text{(normalization of Haar measure)}
\end{aligned}$$

Werner twirl

We used the fact that $\mathbb{F}_{AB}$ is invariant under the unitary $(U \otimes U)^\dagger$:
For all states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$, we have

$$\begin{aligned}(U \otimes U)^\dagger \mathbb{F}_{AB}(U \otimes U)(|\psi\rangle \otimes |\phi\rangle) &= (U \otimes U)^\dagger \mathbb{F}_{AB}(U|\psi\rangle \otimes U|\phi\rangle) \\ &= (U^\dagger \otimes U^\dagger)(U|\phi\rangle \otimes U|\psi\rangle) \\ &= |\phi\rangle \otimes |\psi\rangle \\ &= \mathbb{F}_{AB}(|\psi\rangle \otimes |\phi\rangle),\end{aligned}$$

and thus the invariance property follows.

Second step: Apply to Werner state and use linearity of trace:

$$\mathrm{tr}\left[\mathbb{F}_{AB}\rho_{AB}\right] = \mathrm{tr}\left[\mathbb{F}_{AB}\mathcal{T}(\rho_{AB})\right] = \frac{2-x}{6}\mathrm{tr}\mathbb{F}_{AB} + \frac{2x-1}{6}\mathrm{tr}\mathbb{F}_{AB}^2 = \frac{2-x}{6}2 + \frac{2x-1}{6}4 = x.$$

Note that $\mathrm{tr}\mathbb{F}_{AB} = 2$ (eigenvalues $+1, +1, +1, -1$) and $\mathrm{tr}\mathbb{F}_{AB}^2 = \mathrm{tr}\mathbb{1}_{AB} = 4$.

Entanglement of two-qubit Werner states

Entanglement of two-qubit Werner states

The Werner state $\rho_{AB} = \frac{2-x}{6}\mathbb{1}_{AB} + \frac{2x-1}{6}\mathbb{F}_{AB}$ is entangled if and only if $x < 0$.

Proof.

First step: Use PPT criterion to show that ρ_{AB} is entangled for $x \in [-1, 0)$.

Recall: $\mathbb{F}_{AB} = 2(\Phi^+_{AB})^{T_B} \implies \mathbb{F}^{T_B}_{AB} = 2\Phi^+_{AB}$, and $\mathbb{1}^{T_B}_{AB} = \mathbb{1}_A \otimes \mathbb{1}^T_B = \mathbb{1}_{AB}$.

Taking partial transpose of ρ_{AB} and using linearity:

$$\rho^{T_B}_{AB} = \frac{2-x}{6}\mathbb{1}^{T_B}_{AB} + \frac{2x-1}{6}\mathbb{F}^{T_B}_{AB} = \frac{2-x}{6}\mathbb{1}_{AB} + \frac{2x-1}{3}\Phi^+_{AB} = \begin{pmatrix} z & \cdot & \cdot & \cdot \\ \cdot & y & \cdot & \cdot \\ \cdot & \cdot & y & \cdot \\ \cdot & \cdot & \cdot & y \end{pmatrix}$$

in Bell basis with eigenvalues $y = \frac{2-x}{6}$ and $z = \frac{2-x}{6} + \frac{2x-1}{3}$.

We have $z < 0$ iff $x < 0 \implies \rho_{AB}$ is entangled in this range. □

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Proof.

Second step: ρ_{AB} is separable for $x \geq 0$

For product states: Use *swap trick* $\operatorname{tr}[(X \otimes Y)\mathbb{F}] = \operatorname{tr}(XY)$ (exercise!) to show that

$$\operatorname{tr}[(\omega_A \otimes \sigma_B)\mathbb{F}_{AB}] = \operatorname{tr}(\omega_A \sigma_B) \geq 0.$$

Generalize to separable states $\tau_{AB} = \sum_i p_i \omega_{A,i} \otimes \sigma_{B,i} \implies \operatorname{tr}(\mathbb{F}_{AB}\tau_{AB}) \geq 0$.

Twirling $\mathcal{T}(\cdot) = \int dU\,(U \otimes U)(\cdot)(U \otimes U)^\dagger$ maps separable states to separable states.

Thus, parameter $x = \operatorname{tr}(\mathbb{F}_{AB}\mathcal{T}(\tau_{AB})) = \operatorname{tr}(\mathbb{F}_{AB}\tau_{AB}) \geq 0$ for any twirled separable state.

For fixed $x \geq 0$ it is easy (exercise!) to construct states ω_A and σ_B such that
$x = \operatorname{tr}[\mathbb{F}_{AB}\mathcal{T}(\omega_A \otimes \sigma_B)] = \operatorname{tr}(\omega_A \sigma_B) \quad \implies \quad$ claim follows! ☐

Assertion of the proposition (with different constants) holds for arbitrary d!

Representations

Groups

Disclaimer

Very brief introduction to group theory and representation theory!

More details can be found e.g. in [Goo14] (group theory) or [Ser77; FH13; Eti+11; Tel05] (representation theory).

Definition (Group)

A group $(G, \cdot)$ is a set G together with a binary operation $\cdot : G \times G \to G$ satisfying:

- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$.
- Identity element: there exists an element $e \in G$ with $e \cdot g = g \cdot e = g$ for all $g \in G$.
- Inverse: for all $g \in G$, there exists $h \in G$ with $g \cdot h = h \cdot g = e$. Such an element h is unique; it is called the *inverse* of g,and denoted by g^{-1}.

Note that the identity element e in a group is unique: If e' is another identity element, then $e' = e'e = e$, where the second equality follows since e is also an identity element.

Commutative groups ($gh = hg$ for all $g, h \in G$) are called **abelian**.

The cardinality $|G|$ of a group is called its *order*.

Examples of groups

(i) Let V be a vector space with vector space addition denoted by '+'. Then $(V, +)$ is a group.

(ii) Let $(\mathbb{K}, +, \cdot)$ be a field, then both $(\mathbb{K}, +)$ and $(\mathbb{K}\backslash\{0\}, \cdot)$ are groups.

(iii) The set of bijections from the set $\{1, 2, ..., n\}$ to itself together with function composition is a group called the *symmetric group* S_n.

(iv) The set of invertible linear maps from a vector space V to itself together with function composition forms the *general linear group* $\mathrm{GL}(V)$. If $\dim V = d$, then this group can be identified with the group of $(d \times d)$-invertible matrices.

(v) The group $\mathcal{U}(V) = \{U \in \mathrm{GL}(V) : U^\dagger U = \mathbb{1}\}$ is the unitary group on V. If $\dim V = d$, then this group can be identified with $\mathcal{U}_d$, the group of $(d \times d)$-unitary matrices.

Group homomorphisms

Group homomorphisms are functions between groups that respect the group structure:

Definition (Group homomorphism)

Let $(G, \cdot)$ and $(H, *)$ be two groups. A group homomorphism $\varphi : G \to H$ is a function satisfying, for all $x, y \in G$,

$$\varphi(x \cdot y) = \varphi(x) * \varphi(y).$$

Let $\varphi : G \to H$ be a group homomorphism between groups $(G, \cdot)$ and $(H, *)$.

(i) $\varphi(e_G) = e_H$: for any $g \in G$ we have $\varphi(g) = \varphi(e_G \cdot g) = \varphi(e_G) * \varphi(g)$, and the identity element in H is unique.

(ii) $\varphi(g^{-1}) = \varphi(g)^{-1}$: for any $g \in G$ we have $e_H = \varphi(e_G) = \varphi(g \cdot g^{-1}) = \varphi(g) * \varphi(g^{-1})$, and the inverse of $\varphi(g)$ in H is unique.

Representations

Representation theory is the study of (abstract) groups via their linear action on (concrete) vector spaces.

Definition (Representation of a group)

A representation (φ, V) of a group G on a vector space V over a field $\mathbb{F}$ is a group homomorphism $\varphi : G \to \mathrm{GL}(V)$.

We always have $\varphi(e) = \mathbb{1}_V$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$.

The **dimension** or **degree** of a representation (φ, V) is the dimension of V.

Representation (φ, V) is called **unitary**, if $\varphi(G) \leq \mathcal{U}(V)$ (every representation operator $\varphi(g)$ is unitary).

Representations of finite groups (and *compact* groups, see later) over $\mathbb{C}$ can always be chosen unitary.

Examples of representations

(i) Let V be any vector space, and let G be an arbitrary group. Setting $\varphi(g) = \mathbb{1}_V$ for all $g \in G$ defines the **trivial representation**.

(ii) Let G be a cyclic group of order d generated by g. Let $V = \mathbb{C}^d$ with basis $|0\rangle, |1\rangle, \ldots, |d-1\rangle$. Consider a linear operator X on V defined by $X|i\rangle = |i+1 \bmod d\rangle$ for all i. Then the map $g \mapsto X$ determines a representation (φ, V) of G.

(iii) Another representation (φ', V) of G is defined by the map $g \mapsto Z$, where $Z|j\rangle = \omega^j |j\rangle$ for a primitive d-th root of unity ω.

The two representations in (ii) and (iii) are essentially the same:

Definition (Equivalent representations)

Let G be a group. Two representations (φ, V) and (φ', V') of G are said to be *isomorphic* or *equivalent* if there exists an isomorphism $f : V \to V'$ such that for all $g \in G$

$$\varphi'(g) = f \circ \varphi(g) \circ f^{-1}.$$

Shift matrix $X : |i\rangle \mapsto |[i-1] \bmod d\rangle$ has eigenvalues $e^{2\pi i k/d}$ for $0 \le k \le d-1$.
Diagonalizing unitary U satisfies $\varphi' = U \circ \varphi \circ U^\dagger$.

More representations

Representations that can be defined for any finite group G:

(i) **Trivial representation:** $g \mapsto \mathbb{1}_V$ for some vector space V.

(ii) **Regular representation:** Let $n = |G|$ and $V \cong \mathbb{C}^n$ with basis $\{|g\rangle\}_{g \in G}$.
Linear extension of the map $\varphi(g) \colon |h\rangle \mapsto |gh\rangle$ is called the *regular representation*.
If (ψ, W) is any representation such that there exists $w \in W$ so that $\{\psi(g)(w)\}_{g \in G}$ is a basis of W, then ψ is isomorphic to the regular representation (exercise).

(iii) **Permutation representation:** Let X be a finite set and G be a group acting on X.
Consider free vector space generated by X: $V \cong \mathbb{C}^m$ with $m = |X|$ and basis $\{|x\rangle\}_{x \in X}$.
Linear extension of the map $\varphi(g) \colon |x\rangle \mapsto |gx\rangle$ defines the *permutation representation* of G.

Note that the regular representation of G is the permutation representation of G that results from G acting on itself by left multiplication.

Irreducible representations and decompositions

Invariant subspaces

Crucial concept in representation theory: decompose a representation into ‘building blocks’ (later called irreducible representations).

Invariant subspaces and subrepresentations

Let (φ, V) be a representation of a group G. A subspace $W \subset V$ is called G-*invariant* if

$$\varphi(g)|w\rangle \in W \quad \text{for all } |w\rangle \in W \text{ and } g \in G.$$

The restriction $\varphi|_W$ of φ onto W is called a *subrepresentation*.

Note that $\{0\}$ and V are always invariant subspaces of any representation.

Example: Trivial representation inside regular representation

Let G be a finite group with $n = |G|$ and $(\varphi, \mathbb{C}^n)$ be the regular representation. Define

$$W = \operatorname{span}\left(\sum_{g\in G} |g\rangle\right).$$

Then $(\varphi|_W, W)$ is a subrepresentation of $(\varphi, \mathbb{C}^n)$ equivalent to the trivial representation.

Invariant subspaces

Let (ψ, V) be a representation of a finite group G of degree $m = \dim V$, and let $W \leq V$ be a G-invariant subspace of dimension $k = \dim W$.

Choose a basis $\{w_1, \dots, w_k, w_{k+1}, \dots, w_m\}$ for V such that $W = \operatorname{span}(w_1, \dots, w_k)$, and set $W' = \operatorname{span}(w_{k+1}, \dots, w_m)$ so that $V = W \oplus W'$.

Then every $\psi(g)$ has the following representation matrix with respect to this basis:

$$\psi(g) = \begin{array}{cc} \begin{array}{cc} \quad W \quad & \quad W' \quad \end{array} & \\ \left(\begin{array}{c|c} \psi(g)\big|_W & * \\ \hline 0 & * \end{array}\right) & \begin{array}{c} W \\ W' \end{array} \end{array}$$

We will see later that there always exist a basis for which the top-right block is also zero.

Irreducible representations

For irreducible representations (or *irreps*), $\{0\}$ and V are the only invariant subspaces:

Irreducible representation

A representation (φ, V) of a group G is called *irreducible* if $\{0\}$ and V are the only G-invariant subspaces of V.

A one-dimensional representation is always irreducible, since one-dimensional vector spaces have no non-trivial subspaces.

Example: one-dimensional subrepresentation $W = \mathrm{span}(\sum_{g \in G} |g\rangle)$ of the regular representation is irreducible.

A major goal of representation theory is to find all irreducible representations of a group G over a given field $\mathbb{F}$. This is always possible over $\mathbb{C}$, and can be done efficiently [BR90].

Direct sums of representations

Definition

Let (φ_1, V_1) and (φ_2, V_2) be representations of a group G. Then the vector space direct sum $V_1 \oplus V_2$ affords the of G-representation

$$[(\varphi_1 \oplus \varphi_2)(g)](v_1 \oplus v_2) \coloneqq [\varphi_1(g)](v_1) \oplus [\varphi_2(g)](v_2).$$

This is called the *direct sum* of the representations (φ_1, V_1) and (φ_2, V_2).

The direct sum construction allows us to decompose representations. In finite dimensions this process necessarily terminates.

Invariant complements

Let (φ, V) be a representation of a finite group G for which V is a vector space over a field whose characteristic does not divide the order of G. Then every G-invariant subspace W has a G-invariant complement W', i.e., $V = W \oplus W'$ (as vector spaces and as representations).

Decomposing representations

Invariant complements

Let (φ, V) be a representation of a finite group G for which V is a vector space over a field whose characteristic does not divide the order of G. Then every G-invariant subspace W has a G-invariant complement W', i.e., $V = W \oplus W'$ (as vector spaces and as representations).

Proof sketch.

Let P_W be the projection onto W and define

$$Q_W = \frac{1}{|G|} \sum_{g \in G} \varphi(g) P_W \varphi(g)^{-1}.$$

Then one can check that $\mathrm{im} Q_W = W$ and $\varphi(g) Q_W = Q_W \varphi(g)$ for all $g \in G$. It then follows that $W' := \ker Q_W$ is the desired G-invariant complement (exercise!). ☐

Decomposing representations

There's an alternative proof for representations over $\mathbb{C}$ that constructs the G-invariant complement by redefining the Hilbert space structure:

Alternative proof using Weyl's unitarity trick.

Let (φ, V) be a representation over $\mathbb{C}$ and let $\langle\cdot|\cdot\rangle\colon V \times V \to \mathbb{C}$ be an inner product on V. Define a new inner product by

$$\langle v|w\rangle_G := \frac{1}{|G|}\sum_{g\in G}\langle\varphi(g)v|\varphi(g)w\rangle.$$

Then for every G-invariant subspace W the orthogonal complement $W^\perp$ taken w.r.t. $\langle\cdot|\cdot\rangle_G$ is G-invariant as well, and $V = W \oplus W^\perp$ as representations. Moreover, (φ, V) is a *unitary representation* w.r.t $\langle\cdot|\cdot\rangle_G$, that is, $\varphi(G) \subset U(V)$ and $\varphi(g^{-1}) = \varphi(g)^{-1} = \varphi(g)^\dagger$.

For general unitary representations (φ, V) and an invariant subspace $W \subset V$, the orthogonal complement $W^\perp$ is again invariant. □

Decomposing representations

Definition

A representation is called *completely reducible* if it is equivalent to a direct sum of irreducible representations.

Maschke's theorem

Every finite-dimensional representation of a finite group G over a field with characteristic not dividing $|G|$ is completely reducible.

Proof.

Use induction on $\dim V$ and the preceding proposition. ☐

Maschke's theorem states that for a finite group G and a finite-dimensional representation V of G over $\mathbb{C}$ we can always write $V = V_1 \oplus \cdots \oplus V_m$ with each V_i irreducible. Is this decomposition unique?

The following result called **Schur's lemma** helps us answer this question. It is a basic observation, but incredibly useful in representation theory and its applications.

Schur's lemma

Schur's lemma

Let (φ_1, V_1) and (φ_2, V_2) be irreducible representations of a group G, and let $f : V_1 \to V_2$ be a G-equivariant linear map, that is, $f \circ \varphi_1(g) = \varphi_2(g) \circ f$ for all $g \in G$. Then the following hold:

(i) Either f is invertible (and hence $V_1 \cong V_2$) or $f = 0$.

(ii) If $V_1 = V_2$ is finite-dimensional over an algebraically closed field $\mathbb{F}$ (for example $\mathbb{F} = \mathbb{C}$), then $f = \lambda \mathbb{1}_{V_1}$ for some $\lambda \in \mathbb{F}$.

Proof.

(i) Suppose $f \neq 0$. Then $\ker f \neq V_1$ is a G-invariant subspace of V_1 (see exercises), so $\ker f = \{0\}$ by irreducibility of V_1. Likewise, $\operatorname{im} f \neq \{0\}$ is a G-invariant subspace of V_2 (see exercises), so $\operatorname{im} f = V_2$ by irreducibility of V_2. This proves that f is invertible.

(ii) $\mathbb{F}$ being algebraically closed guarantees that the linear map f has an eigenvalue, say $\lambda \in \mathbb{F}$. The map $f' = f - \lambda \mathbb{1}_{V_1}$ is G-equivariant, and it is not invertible since its kernel has a non-zero eigenvector of f. By (*i*), it follows that $f' = 0$, and so $f = \lambda \mathbb{1}_{V_1}$. ☐

Applications of Schur's lemma

A consequence of Schur's lemma is the following useful result.

Irreps of abelian groups are 1-dimensional

Let G be an abelian group. Then any complex irreducible representation of G is one-dimensional.

Proof.

Let (φ, V) be an irreducible representation of G. Since G is abelian, we have

$$\varphi(g)\varphi(h) = \varphi(gh) = \varphi(hg) = \varphi(h)\varphi(g)$$

for all $g, h \in G$. It follows from Schur's Lemma that $\varphi(g) = \lambda_g \mathbb{1}_V$ for every $g \in G$, and so $\varphi(g)v = \lambda_g v$ for all $v \in V$, that is, every non-zero $v \in V$ spans a one-dimensional G-invariant subspace of V. Since φ is irreducible, V must be one-dimensional itself. □

Isotypical decomposition

Schur's lemma lets us decompose a representation into groups of inequivalent irreps:

Definition (Isotypical decomposition)

Let (φ, V) be a finite-dimensional representation of a finite group G over $\mathbb{C}$. Consider a decomposition $V \cong \bigoplus_k V_k$ and $\varphi \cong \bigoplus_k \varphi_k$ with the following properties:

- Each (φ_k, V_k) is the direct sum of n_k copies of an irrep (ψ_k, W_k) of G:

$$V_k = W_k \oplus \cdots \oplus W_k = W_k^{\oplus n_k} \cong W_k \otimes \mathbb{C}^{n_k}$$

$$\varphi_k = \psi_k \oplus \ldots \psi_k = \psi_k \otimes \mathrm{id}_{\mathbb{C}^{n_k}}.$$

- The different W_k are inequivalent, that is, $W_k \ncong W_{k'}$ for $k \neq k'$.

Then V_k is called an *isotypical component*, and

$$V = \bigoplus_k V_k = \bigoplus_k W_k^{n_k} = \bigoplus_k W_k \otimes \mathbb{C}^{n_k}$$

(with $\varphi = \bigoplus_k \varphi_k$) is called *isotypical decomposition.*

Isotypical decomposition

An application of Schur's lemma shows (see [Ser77; Tel05] for a proof):

Uniqueness of isotypical decomposition

The decomposition $V = \bigoplus_k V_k$ of a representation V into isotypical components V_k exists by Maschke's theorem and is unique, and so are the multiplicities n_k of W_k in V_k.

Another application of Schur's Lemma: symmetrizing an arbitrary operator.

Symmetrizing a representation

Let (φ, V) be a representation of a finite group G with isotypical decomposition

$$V = \bigoplus_k W_k \otimes \mathbb{C}^{n_k}.$$

Then for an arbitrary operator $X \in \mathcal{L}(V)$, we have

$$\frac{1}{|G|} \sum_{g \in G} \varphi(g) X \varphi(g)^{-1} = \bigoplus_k \mathbb{1}_{W_k} \otimes X_k,$$

where the $X_k \in \mathcal{L}(\mathbb{C}^{n_k})$ are suitable operators acting on the multiplicity spaces $\mathbb{C}^{n_k}$.

Applications of Schur’s Lemma

Symmetrizing with respect to an irrep

Pauli group

Let ρ be a qubit density matrix. Then

$$\frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) = \frac{1}{2}\mathbb{1}.$$

Check explicitly **or better**, check that Pauli matrices $\mathbb{1}, X, Y, Z$ generate an irreducible representation of the Pauli group, and use Schur's Lemma (exercise!).

Full unitary group

Same idea for the full unitary group: $\int_{\mathcal{U}_2} \mathrm{d}U\, U\rho U^\dagger = \frac{1}{2}\mathbb{1}$.

1-designs

For every (unitary) irrep $(\varphi, \mathbb{C}^d)$ of a finite group G,

$$\frac{1}{|G|}\sum_{g\in G}\varphi(g)X\varphi(g)^\dagger = \int_{\mathcal{U}_d} \mathrm{d}U\, UXU^\dagger = \frac{\mathrm{tr}(X)}{d}\mathbb{1}.$$

Natural permutation representation

Fix an orthonormal basis $\{|1\rangle, \dots, |n\rangle\}$ of $\mathbb{C}^n$.

Let S_n act by permuting basis vectors:

$$\varphi : S_n \to \mathrm{GL}(\mathbb{C}^n), \quad \pi \mapsto (\varphi(\pi) : |i\rangle \mapsto |\pi(i)\rangle).$$

Representation matrices $\varphi(\pi)$ are **unitary permutation matrices**.

Vector $|v\rangle = \sum_{i=1}^{n} |i\rangle$ spans 1-dimensional subspace $W_{\mathrm{triv}} \leq \mathbb{C}^n$ on which S_n acts **trivially**:

$$\varphi(\pi)|v\rangle = |v\rangle \quad \text{for every } \pi \in S_n.$$

W_{triv} is an irrep, and the permutation representation decomposes as:

$$\mathbb{C}^n \cong W_{\mathrm{triv}} \oplus W_{\mathrm{st}}$$
$$\varphi \cong \varphi_{\mathrm{triv}} \oplus \varphi_{\mathrm{st}}.$$

How to determine W_{st}?

Determining invariant complement

Let $P_{W_{\text{triv}}} = |v\rangle\langle v|$ be the (orthogonal) projection onto W_{triv}.

Since $|v\rangle$ is S_n-invariant, this is already an S_n-invariant projection:

$$\varphi(\pi) P_{W_{\text{triv}}} \varphi(\pi)^{\dagger} = P_{W_{\text{triv}}} \quad \text{for all } \pi \in S_n.$$

Invariant complement

We define $W_{\text{st}} := \ker P_{W_{\text{triv}}} = \left(P_{W_{\text{triv}}}\right)^{\perp}$, which is given by

$$W_{\text{st}} = \left\{ \sum_{i=1}^{n} x_i |i\rangle : x_1 + \cdots + x_n = 0 \right\}.$$

Standard representation

W_{st} is called the standard representation of S_n. It is an irreducible representation of degree $\dim W_{\text{st}} = n - 1$.

Proof: Exercise.

The case $n = 2$

$S_2 = \{(), (12)\}$ with representation matrices

$$\varphi(()) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \varphi((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Decomposition of representation spaces:

$$V \cong W_{\mathrm{triv}} \oplus W_{\mathrm{st}} \qquad \text{where} \quad \begin{aligned} W_{\mathrm{triv}} &= \mathrm{span}(|1\rangle + |2\rangle) \\ W_{\mathrm{st}} &= \mathrm{span}(|1\rangle - |2\rangle). \end{aligned}$$

Transposition (12) acts on W_{st} by multiplying with $\mathrm{sgn}((12)) = -1$:

$$\varphi((12))(|1\rangle - |2\rangle) = -(|1\rangle - |2\rangle).$$

Decomposition of representations:

$$\varphi(()) \cong \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) \qquad \varphi((12)) \cong \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)$$

Symmetrizing operators

Decomposition $V \cong W_{\text{triv}} \oplus W_{\text{st}}$ and $\varphi(()) \cong \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$, $\varphi((12)) \cong \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)$.

In the basis $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle), \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$, an arbitrary operator $O \in \mathcal{L}(\mathbb{C}^2)$ can be written as

$$O = \left(\begin{array}{c|c} O_{W_{\text{triv}} \to W_{\text{triv}}} & O_{W_{\text{st}} \to W_{\text{triv}}} \\ \hline O_{W_{\text{triv}} \to W_{\text{st}}} & O_{W_{\text{st}} \to W_{\text{st}}} \end{array}\right).$$

Symmetrizing an operator $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives:

$$\overline{X} = \frac{1}{2}\left(X + \varphi((12))X\varphi((12))^{\dagger}\right) = \frac{1}{2}\begin{pmatrix} a+d & b+c \\ b+c & a+d \end{pmatrix} \cong \frac{1}{2}\left(\begin{array}{c|c} a+b+c+d & 0 \\ \hline 0 & a-b-c+d \end{array}\right)$$

$$= \frac{1}{2}\left(\begin{array}{c|c} (a+b+c+d)\mathbb{1}_{W_{\text{triv}}} & 0 \\ \hline 0 & (a-b-c+d)\mathbb{1}_{W_{\text{st}}} \end{array}\right).$$

The case $n = 3$

We choose the orthonormal basis

$$|v_1\rangle = \tfrac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \qquad |v_2\rangle = \tfrac{1}{\sqrt{6}}(2|1\rangle - |2\rangle - |3\rangle) \qquad |v_3\rangle = \tfrac{1}{\sqrt{2}}(|2\rangle - |3\rangle),$$

so that $W_{\mathrm{triv}} = \mathrm{span}(|v_1\rangle)$ and $W_{\mathrm{st}} = \mathrm{span}(|v_2\rangle, |v_3\rangle)$.

Symmetrization of an arbitrary operator X:

$$\overline{X} = \frac{1}{3!}\sum_{\pi\in S_3} \varphi(\pi) X \varphi(\pi)^\dagger = \begin{pmatrix} z & y & y \\ y & z & y \\ y & y & z \end{pmatrix} \cong \begin{pmatrix} z+2y & 0 & 0 \\ 0 & z-y & 0 \\ 0 & 0 & z-y \end{pmatrix}$$

$$= \left(\begin{array}{c|cc} (z+2y)\mathbb{1}_{W_{\mathrm{triv}}} & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$(z-y)\mathbb{1}_{W_{\mathrm{st}}}$}} \\ 0 & & \end{array}\right)$$

$$= \left(\begin{array}{c|c} \overline{X}_{W_{\mathrm{triv}}\to W_{\mathrm{triv}}} & \overline{X}_{W_{\mathrm{st}}\to W_{\mathrm{triv}}} \\ \hline \overline{X}_{W_{\mathrm{triv}}\to W_{\mathrm{st}}} & \overline{X}_{W_{\mathrm{st}}\to W_{\mathrm{st}}} \end{array}\right).$$

Limitations of Schur's Lemma: Multiplicities

Recall tensor representation of S_2 on $\mathbb{C}^2 \otimes \mathbb{C}^2$ (where $\mathbb{F} \equiv \varphi((12))$):

$$\mathbb{F}(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle.$$

Orthonormal eigenbasis of $\mathbb{F}$ is given by the four Bell states:

$$\begin{aligned} |\Phi^+\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & \qquad |\Psi^+\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\Phi^-\rangle &= \tfrac{1}{\sqrt{2}}(|00\rangle - |11\rangle) & \qquad |\Psi^-\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned}$$

These span the following irreducible representations:

$$\begin{aligned} W_{\text{triv}} &\cong \operatorname{span}(|\Phi^+\rangle) \cong \operatorname{span}(|\Phi^-\rangle) \cong \operatorname{span}(|\Psi^+\rangle) \\ W_{\text{sgn}} &\cong \operatorname{span}(|\Psi^-\rangle). \end{aligned}$$

Isotypical decomposition:

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong V_{\text{triv}} \oplus V_{\text{sgn}} \qquad \text{where} \quad \begin{aligned} V_{\text{triv}} &= W_{\text{triv}} \oplus W_{\text{triv}} \oplus W_{\text{triv}} \cong W_{\text{triv}} \otimes \mathbb{C}^3 \\ V_{\text{sgn}} &\cong W_{\text{sgn}}. \end{aligned}$$

Limitations of Schur's Lemma: Multiplicities

Symmetrizing an operator with respect to tensor representation:

$$\begin{aligned}\overline{X} = \frac{1}{2}(X + \mathbb{F}X\mathbb{F}^\dagger) &\cong \left(\begin{array}{c|c} \overline{X}_{V_{\text{triv}}\to V_{\text{triv}}} & 0 \\ \hline 0 & x_{\text{sgn}}\mathbb{1}_{W_{\text{sgn}}} \end{array}\right) \\ &\cong \left(\mathbb{1}_{W_{\text{triv}}} \otimes X_{\text{triv}}\right) \oplus x_{\text{sgn}}\mathbb{1}_{W_{\text{sgn}}}.\end{aligned}$$

Note that $\dim W_{\text{triv}} = 1$ and hence $\mathbb{1}_{W_{\text{triv}}} = 1$.

In the above, the top-left block of the symmetrized operator is not proportional to the identity, since it is not just a map from an irrep into itself.

Tensor and dual representations, hom spaces

Tensor representation

We now discuss how to produce new representations from given ones.

Definition

Let (φ, V) and (ψ, W) be representations of a group G. Then $(\varphi \otimes \psi)(g) \coloneqq \varphi(g) \otimes \psi(g)$ defines a representation on $V \otimes W$ called the *tensor representation*.

$V \otimes W$ is in general *reducible*, even for irreducible V, W:

Symmetric and antisymmetric square

Let (φ, V) be a representation of a group G and consider the tensor representation $(\varphi\otimes, \varphi, V \otimes V)$. The swap operator $\mathbb{F}$ commutes with the action of $\psi \otimes \psi$, and we thus have the decomposition $V \otimes V = \mathrm{Sym}^2(V) \oplus \Lambda^2(V)$, where

$$\begin{aligned} \mathrm{Sym}^2(V) &\coloneqq \{|z\rangle \in V \otimes V : \mathbb{F}|z\rangle = |z\rangle\} \\ \Lambda^2(V) &\coloneqq \{|z\rangle \in V \otimes V : \mathbb{F}|z\rangle = -|z\rangle\}. \end{aligned}$$

Symmetric and antisymmetric square

If $\dim V = d$ and $\{|e_i\rangle\}_{i=1}^{d}$ is a basis for V, then these two subspaces can be constructed as

$$\mathrm{Sym}^2(V) = \mathrm{span}\{|e_i\rangle \otimes |e_j\rangle + |e_j\rangle \otimes |e_i\rangle : 1 \leq i \leq j \leq d\}$$
$$\Lambda^2(V) = \mathrm{span}\{|e_i\rangle \otimes |e_j\rangle - |e_j\rangle \otimes |e_i\rangle : 1 \leq i < j \leq d\}.$$

In fact, these two sets of spanning vectors form bases for $\mathrm{Sym}^2(V)$ and $\Lambda^2(V)$, and hence

$$\dim(\mathrm{Sym}^2(V)) = \frac{d(d+1)}{2} \qquad \dim(\Lambda^2(V)) = \frac{d(d-1)}{2}.$$

$\mathrm{Sym}^2(V)$ and $\Lambda^2(V)$ are both G-invariant subspaces, and thus representations of G, called the *symmetric* and *antisymmetric square*, respectively.

Setting $V = \mathbb{C}^2$, this construction gives the symmetric and antisymmetric subspaces.

This example shows that $V \otimes V$ is reducible whenever V has degree at least 2.

Dual and hom-space representations

Definition

Let (φ, V) be a representation of G. Let V^* be the dual space of V consisting of the vector space of linear maps from V to $\mathbb{C}$. The *dual* representation (φ^*, V^*) is defined as

$$\varphi^*(g)(L) \coloneqq L \circ \varphi(g)^{-1} \textbf{ for } g \in G \textbf{ and } L \in V^* \textbf{.}$$

The dual representation satisfies for all $g \in G$, $|v\rangle \in V$, and $\langle w| \in V^*$ that $\varphi^*(g) = \varphi(g^{-1})^T$.

Moreover, (φ^*, V^*) is irreducible iff (φ, V) is.

If (φ, V) is unitary, then $\varphi^*(g) = \overline{\varphi(g)}$, that is, $\varphi^*(g)$ is the complex conjugate of $\varphi(g)$.

Definition (Hom-space representation)

Let (φ, V) and (ψ, W) be two representations of a group G. Then G acts on $\mathrm{Hom}(V, W)$ by sending $f : V \to W$ to $\psi(g) \circ f \circ \varphi(g)^{-1}$, which turns $\mathrm{Hom}(V, W)$ into a representation of G.

Note that setting $W = \mathbb{C}$ and ψ the trivial representation of G in the definition above recovers the dual representation of G.

Properties of the hom-space representation

1. $\mathrm{Hom}(V, W) \cong V^* \otimes W$ as vector spaces and representations (exercise).
2. The set of vectors in V invariant under the action of G is denoted as

$$V^G \coloneqq \{|v\rangle \in V : \varphi(g)|v\rangle = |v\rangle \text{ for all } g \in G\}.$$

   With this notation we have $\mathrm{Hom}_G(V, W) \coloneqq \mathrm{Hom}(V, W)^G = (V^* \otimes W)^G$.
   An element $f : V \to W$ of $\mathrm{Hom}_G(V, W)$ is called an *intertwiner* of the representations (φ, V) and (ψ, W), satisfying $f \circ \varphi(g) = \psi(g) \circ f$ for all $g \in G$.
3. If $V = \bigoplus_i V_i$ is an isotypical decomposition of a representation V of G with components $V_i = W_i^{n_i}$ for pairwise inequivalent irreducible representations W_i of G, then

$$n_i = \dim \mathrm{Hom}_G(W_i, V) = \dim(W_i^* \otimes V)^G.$$

   See [Ser77, Ex. 2.8] for a proof.

Group algebra and characters

Group algebra

Regular representation (φ_R, V) of a group G: vector space V generated by $\{|g\rangle\}_{g\in G}$ together with group action $\varphi_R(g)|h\rangle = |gh\rangle$.

The group multiplication endows V with the structure of an **algebra**:

$$\left[\sum_{g\in G} c_g |g\rangle\right] \cdot \left[\sum_{h\in G} d_h |h\rangle\right] = \sum_{g,h\in G} c_g d_h |gh\rangle = \sum_{g\in G} f_g |g\rangle \quad \text{with} \quad f_g = \sum_{h\in G} c_{gh^{-1}} d_h.$$

This multiplication on V is associative, has the group identity element e as the multiplicative identity, and satisfies distributivity over addition.

Group algebra

$\mathbb{C}[G] \coloneqq (V, +, \cdot)$ is called the *group algebra*.

Class functions and characters

Elements $\sum_{g\in G} c_g \,|g\rangle$ of $\mathbb{C}[G]$ can be understood as functions $f : G \to \mathbb{C}$ via $g \mapsto c_g$

A function $f : G \to \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes of G:

$$f(g) = f(hgh^{-1}) \quad \text{for all } g, h \in G.$$

Character of a representation

Let (φ, V) be a representation of G. The *character* $\chi = \chi_V$ of (φ, V) is the class function defined by $\chi(g) = \mathrm{tr}(\varphi(g))$.

A character is indeed a class function: for all $g, h \in G$,

$$\begin{aligned} \chi(hgh^{-1}) &= \mathrm{tr}\left[\varphi(hgh^{-1})\right] \\ &= \mathrm{tr}\left[\varphi(h)\varphi(g)\varphi(h)^{-1}\right] && \text{since } \varphi(xy) = \varphi(x)\varphi(y) \\ &= \mathrm{tr}\,[\varphi(g)] = \chi(g) && \text{by cyclicity of the trace.} \end{aligned}$$

Properties of characters

Let (φ, V) and (ψ, W) be representations of a group G with identity element e, and denote by χ_V and χ_W the associated characters.

(i) $\chi_V(e) = \mathrm{tr}(\mathbb{1}_V) = \dim V$ is the degree of the representation (φ, V).

(ii) If (φ, V) is unitary, then $\chi(g^{-1}) = \overline{\chi(g)}$.

(iii) $\chi_{V \oplus W} = \chi_V + \chi_W$.

(iv) $\chi_{V \otimes W} = \chi_V \chi_W$.

The group algebra $\mathbb{C}[G]$ has a natural inner product structure: For $x = \sum_{g \in G} x_g |g\rangle$ and $y = \sum_{g \in G} y_g |g\rangle$ in $\mathbb{C}[G]$, we define

$$(x, y) \coloneqq \frac{1}{|G|} \sum_{g \in G} \overline{x_g} y_g.$$

Character theory

Orthogonality of characters

Let W_i for $i = 1, \ldots, k$ be pairwise inequivalent irreducible representations of a group G, and denote by χ_i the corresponding characters. Then $(\chi_i, \chi_j) = \delta_{ij}$, and $\{\chi_i\}_{i=1}^{k}$ is an orthonormal basis of the set of class functions.

Proof in [Ser77; Tel05].

Properties of characters

Let (φ, V) be an arbitrary representation of a group G, and let (ψ, W) be an irreducible representation of G.

(i) The multiplicity of W in the isotypical decomposition V is $\left(\chi_V, \chi_W\right)$.
(ii) V is irreducible iff $\left(\chi_V, \chi_V\right) = 1$.
(iii) Two representations are isomorphic iff they have the same character.
(iv) The number of distinct (i.e., pairwise inequivalent) irreducible representations of a finite group G is equal to the number of conjugacy classes of G.

More character theory

(i) The multiplicity of any irreducible representation in the regular representation of a group G is equal to its dimension.

(ii) Let $W_1, \ldots, W_k$ be a complete list of irreducible representations of G. Then every W_i appears in the regular representation, and $\sum_{i=1}^{k} (\dim W_i)^2 = |G|$.

Projection formula

Let (φ, V) be a representation of G, and let W be a fixed irreducible representation of G with character χ_W. Then the projection onto the isotypical component of W in V is given by the formula

$$P_W = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \varphi(g).$$

In particular, let $\chi_{\text{triv}} : g \mapsto 1$ for all $g \in G$ be the character of the trivial representation. Then $P = \frac{1}{|G|} \sum_{g \in G} \varphi(g)$ projects onto $V^G = \{|v\rangle \in V : \varphi(g)|v\rangle = |v\rangle \text{ for all } g \in G\}$ consisting of $(\chi_{\text{triv}}, \chi_V)$-many copies of the trivial representation.

Finite and compact groups

Topological groups

Goal: Extend some of the results for representations of finite groups (decompositions, character theory) to certain infinite groups that “behave” like finite groups.

Definition

A *topological group* is a group G endowed with a topology such that group multiplication and inversion are continuous. A *compact group* is a topological group that is compact, that is, every open cover of G has a finite subcover. Closed subgroups of a compact group are also compact groups.

Definition

A *representation* (φ, V) of a topological group G on a normed, finite-dimensional vector space V is a continuous group homomorphism $\varphi : G \to \mathrm{GL}(V)$.

Haar measure

Crucial ingredient for representation theory of finite groups so far:

$$\textbf{averaging operation} \quad \frac{1}{|G|}\sum_{g\in G} f(g).$$

This can be understood as integrating f against the uniform "point measure" $\frac{1}{|G|}\delta_g$.

For **compact groups** a (unique) uniform probability measure also exists:

Haar measure

Let G be a compact group. There exists a unique measure $\mathrm{d}g$ on G, called the *Haar measure*, satisfying the following properties:

1. Invariance: for every continuous function $f: G \to \mathbb{C}$ and every $h \in G$,

$$\int_G f(g)\mathrm{d}g = \int_G f(gh)\mathrm{d}g = \int_G f(hg)\mathrm{d}g.$$

2. Normalization: $\int_G 1\mathrm{d}g = 1$.

Haar measure

Finite groups are compact

Every finite group with the discrete topology is a compact group. In this setting, the Haar measure is equal to the counting measure, and we have $\int_G \mathrm{d}g \approx \frac{1}{|G|}\sum_{g\in G}$.

Haar measure on U_1

The circle group $\mathcal{U}_1 = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{\exp(\mathrm{i}\theta) : \theta \in [0, 2\pi)\}$ has Haar measure $\mathrm{d}g = \frac{1}{2\pi}\mathrm{d}\theta$.

General formulas for Haar integration are typically cumbersome:

> *"It is not practical to give an explicit formula for integrating a general function on a group such as $\mathcal{U}_n$, for there are no convenient coordinates to use."*
>
> *— G. Segal, [CSM95, Lie Groups, Ch. 7]*

In applications, one typically avoids explicit calculations involving Haar integrals, and instead uses the invariance property together with Schur's Lemma.

Representation theory of compact groups

Using the Haar measure, one can prove analogous statements about finite-dimensional representations of compact groups, e.g.:

1. Every G-invariant subspace has a G-invariant complement.
2. Every representation over $\mathbb{C}$ decomposes as a sum of irreducible representations.
3. Most aspects of character theory also carry over to the compact case (note however that if G is an infinite compact group, then expressions involving $|G|$ may no longer be valid).

The regular representation of a compact group G is defined as the Hilbert space $L^2(G)$ of square integrable functions on G, with the action of G given by

$$\varphi(g)(f)\colon\ h \mapsto f(g^{-1}h).$$

If $|G| = \infty$, then $\dim L^2(G) = \infty$.

Regular representation of compact groups

We have the following theorem about the decomposition of the regular representation for compact groups.

Regular representation of compact groups

Let G be a compact group.

(i) The linear span of all matrix coefficients of the irreducible unitary representations of G is dense in $L^2(G)$.

(ii) Every irreducible unitary representation of G is finite-dimensional.

(iii) The regular representation (which has infinite dimension if G is not finite) $L^2(G)$ decomposes into a direct sum of the irreducible unitary representations of G, each occurring with multiplicity equal to its dimension. The matrix coefficients of the complete set of irreps form an orthonormal basis of $L^2(G)$.

See [Kna16, Thm. 1.12] for a proof.

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