NPT Bound Entanglement and Werner States

Math 595 QI Presentation

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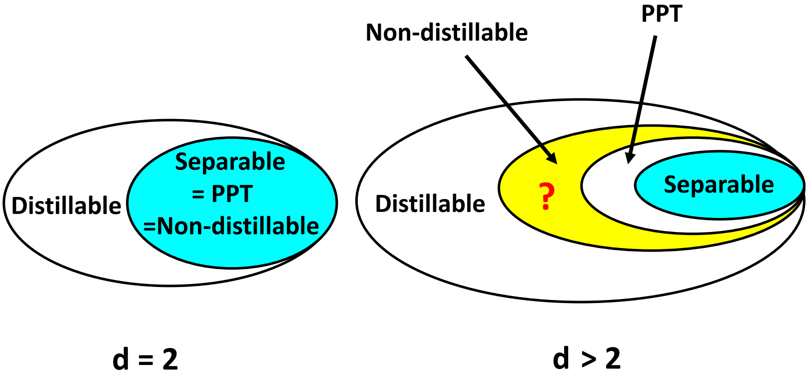
Outline

Problem of interest

Problem of interest: distillation of entanglement of mixed states (as opposed to pure states) at higher dimensional ($d \geq 2$) compound systems (here we care about bipartite systems)

$$\mathscr{H} = \mathscr{H}_d \otimes \mathscr{H}_d$$



PPT Criterion and Entanglement

- The PPT criterion is generally a necessary condition for separability
- PPT: A state ρ_{AB} is called PPT if $\rho_{AB}^{T_B} \geq 0$ (or equivalently, $(\rho_{AB}^{T_A} \geq 0)$
- PPT criterion: Any state ρ_{AB} with negative partial transpose (NPT), $\rho_{AB}^{T_B} \ngeq 0$, is entangled.

Local Operators and Classical Communications (LOCC)

- A local (product) operation is performed on part of the system, and where the result of that operation is "communicated" classically to another part where usually another local operation is performed conditioned on the information received.
- In general, all separable states (and only these) can be prepared from a product states with LOCC operations alone.
- While LOCC cannot generate entangled states out of product states, they can be used to transform entangled states into other entangled states.

Entangled $U \otimes U^*$ invariant states

$$\rho \to U \otimes U^* \rho \, U^\dagger \otimes U^{*\dagger}$$

Isotropic states have entanglement structures that are much easier to understand than that of general bipartite density operators.

General isotropic state:

$$\rho_\alpha = (1-\alpha)\frac{I}{N^2} + \alpha P_+$$

Maximally mixed and maximally entangled state

- Maximally mixed state: $\rho_{max-mixed} = \frac{1}{d} I_d$.
  - all pure states have equal probability
  - no information about any preferred basis or correlations
- Maximally entangled state: $P_+ = |\psi_+\rangle \langle\psi_+|$ with $\psi_+ = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle$.
  - each single qubit is locally maximally mixed
  - for $d = 2$, $\rho_A = Tr_B(P_+) = \frac{1}{2} I_2,\ \rho_B = \frac{1}{2} I_2$.
  - maximally entangled states are optimal resources, and they have strongest possible quantum correlations.

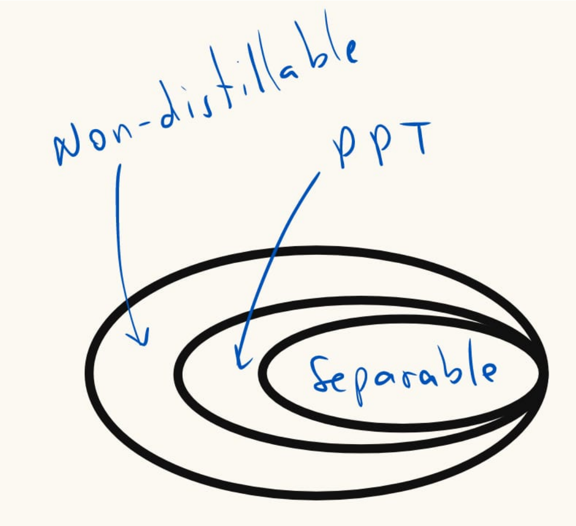
Distillable states and LOCC for entanglement distillation

- Distillable state: a state from which you can asymptotically obtain maximally entangled pairs at a nonzero rate using LOCC on many copies
- Distillable Entanglement: using LOCC (filters, XOR gates, post-selection, etc) to covert many noisy pairs ρ into fewer but more entangled pairs, iteratively increasing fidelity to a Bell (or maximally entangled) state.

Can any entangled state be distilled?

- Can any entangled state be distilled? Answer: No
- As we know, PPT is only a necessary condition for separability, i.e., PPT states can be entangled
- In fact, entangled PPT states are non-distillable
- Entangled states that are non-distillable are called *bound entangled states* $\sim$ low entanglement

Entangled PPT states are non-distillable (Bound entanglement)



Does there exist any NPT state that is bound entangled?

- If entangled PPT states are non-distillable, the question becomes: is there any NPT state that is bound entangled, i.e., non-distillable?
- Horodecki 1998 simplified the question by introducing a new criterion of separability

CP maps

- CP maps are completely positive maps of density matrices
  - Positive: If $\rho \geq 0 \implies \Lambda(\rho) \geq 0$
  - Λ_A, Λ_B are Completely positive if they are Positive and $\Lambda_A \otimes \Lambda_B$ is also Positive
    - Equivalently: Λ is CP if $\Lambda \otimes I$ is positive

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- Positive but not-CP maps provide a necessary condition of separability. Indeed, for any separable state $\rho = \sum_i p_i \, \rho_A^i \otimes \rho_B^i$:

$$(I \otimes \Lambda)(\rho) = (I \otimes \Lambda)\left(\sum_i p_i \, \rho_A^i \otimes \rho_B^i\right) = \sum_i p_i \, \rho_A^i \otimes \Lambda(\rho_B^i) \geq 0 \tag{1}$$

  - Even though Λ is not CP, its mapping of separable states is always positive. Therefore, operators violating the inequality must be entangled
- We already know one such map: **Transpose** is a P but not-CP map $\rightarrow$ PPT criterion

The Reduction Criterion (RC) of separability

- Consider the map: $\Lambda_{RC}(\rho) = I \operatorname{Tr} \rho - \rho$
- Λ_{RC} is a positive map: the trace of ρ (even if ρ is not normalized) is greater than any eigenvalue of ρ (because $\rho \geq 0$)
- Λ_{RC} can be written as: $\Lambda_{RC} = \Lambda_{CP} T \implies$ it is not CP $\implies$ provides a criterion of separability: $(I \otimes \Lambda_{RC})(\rho) \geq 0$ for any separable ρ
- The criterion reads:

$$\boxed{\rho_A \otimes I - \rho \geq 0},$$

where $\rho_A = Tr_B \rho$, $\forall$ separable ρ.

PPT is an stronger criterion

Theorem

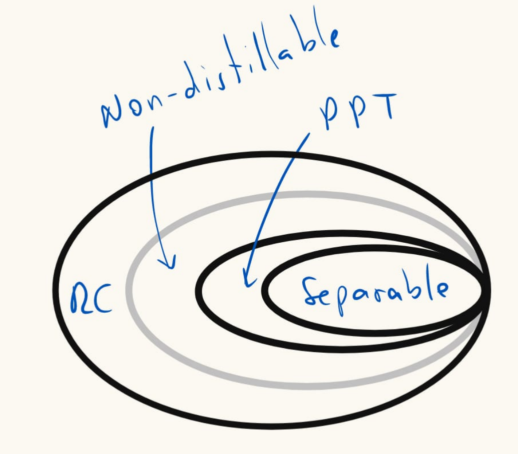
Violating Reduction Criterion $\implies$ *Violating PPT Criterion*

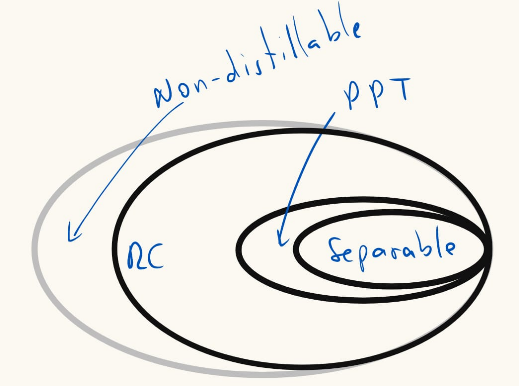
Proof.

- Let us prove the converse. Suppose ρ is a PPT state, i.e., $\rho^{T_B} \geq 0 \iff \sigma \equiv (I \otimes T)(\rho) \geq 0$.
- Recall: $\Lambda_{RC} = \Lambda_{CP} T$, and Λ_{RC} is Positive but not-CP.
- Then, $(I \otimes \Lambda_{RC})(\rho) = (I \otimes \Lambda_{CP} T)(\rho) = [(I \otimes \Lambda_{CP})(I \otimes T)](\rho) = (I \otimes \Lambda_{CP})[(I \otimes T)(\rho)] = (I \otimes \Lambda_{CP})(\sigma) \geq 0$, where the last inequality follows from the definition of CP maps.

□

PPT is an stronger criterion





Violate RC $\implies$ Distillability—Outline

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- Twirling: Want to write ρ' in terms of P_+ keeping the fidelity constant
  - Naturally arises the twirling operator given by $U \otimes U^*$
  - The output is an isotropic state, $(1-\alpha)\frac{I}{N} + \alpha P_+$, with fidelity $F > 1/N$
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  - Isotropic states are entangled precisely $\iff$ $F > 1/N$
- Distill entanglement $\to$ Recover P_+
  - Isotropic states are very convenient: all that is needed is to increase α
  - Use a distillation protocol that increases α
  - Apply recursively: $\alpha \to 1$ as $N \to \infty$ (number of copies)
- QED: Violate RC $\implies$ Distillability

Violate RC $\implies$ Distillability—Filtering

- Equivalent criterion to RC: $\forall \rho$ separable:

$$\rho_A \otimes I - \rho \geq 0 \iff \langle\psi|\, \rho_A \otimes I - \rho\, |\psi\rangle \geq 0\ \forall\psi$$

$$\mathrm{Tr}\left[(\rho_A \otimes I)\,|\psi\rangle\langle\psi|\right] - \mathrm{Tr}(\rho\,|\psi\rangle\langle\psi|) \geq 0 \iff \mathrm{Tr}(\rho\, P_\psi) \ \leq\ \mathrm{Tr}\Big(\rho_A\, \rho_A^\psi\Big)$$

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- Suppose ρ violates this criterion $\implies$:

$$\mathrm{Tr}\,(\rho\, P_\psi) > \mathrm{Tr}\,\left(\rho_A\, \rho_A^\psi\right), \text{ for some } |\psi\rangle = \sum_{m,n} a_{m,n}\, |m\rangle \otimes |n\rangle$$

- Note $|\psi\rangle = (A \otimes I)\, |\psi_+\rangle$ with $\langle m|\, A\, |n\rangle = a_{mn}\sqrt{N}$
- $\rho_A^\psi \equiv \mathrm{Tr}_B(|\psi\rangle\, \langle\psi|) = \mathrm{Tr}_B\left[(A \otimes I)\, |\psi_+\rangle\, \langle\psi_+|\, (A^\dagger \otimes I)\right] = AA^\dagger \frac{1}{N}$

Violate RC $\implies$ Distillability—Filtering

- Recall: $\mathrm{Tr}\,(\rho\, P_\psi) > \mathrm{Tr}\left(\rho_A\, \rho_A^\psi\right) \quad |\psi\rangle = (A \otimes I)\,|\psi_+\rangle \quad \rho_A^\psi = AA^\dagger \frac{1}{N}$
- Consider the new *filtered* state: $\rho' = \frac{(A^\dagger \otimes I)\rho(A \otimes I)}{\mathrm{Tr}\left[\rho(AA^\dagger \otimes I)\right]} \equiv \frac{(A^\dagger \otimes I)\rho(A \otimes I)}{C}$
- Then:

$$\mathrm{Tr}(\rho' P_+) > \frac{1}{N}$$

Violate RC $\implies$ Distillability—Distillation protocol overview

1. Distillation:
   - generalized twirling procedure which would leave the state P_+ unchanged. (which is the entangled part)
     - Twirling operator $\int U \otimes U^* \rho U^\dagger \otimes U^{*\dagger} = \rho_\alpha \equiv (1-\alpha)\frac{I}{N} + \alpha P_+$, with $\frac{-1}{N^2-1} \leq \alpha \leq 1$.
     - after twirling the fidelity is the same
   - XOR gates: if target outcomes are equal, we obtain it in state ρ'_α where $\alpha' > \alpha \to$ the entanglement fraction increases if the initial fraction was greater than $1/N$. i.e., a nonzero asymptotic yiled of distilled pure entanglement
2. Recurrence protocol: get many copies of ρ_α, use LOCC on the ones that we keep, then $\alpha' > \alpha$.

Violate RC $\implies$ Distillability—Recurrence protocol for Entanglement Distillation

$$U_{XOR^N} |k\rangle |l\rangle = |k\rangle |l \oplus k\rangle$$

where $k \oplus l = (k + l)\ mod\ N$. $|k\rangle$ and $|l\rangle$ states describe the source and target systems.

- Two input pairs are twirled i.e. each of them is subjected to random bilateral rotation of type $U \otimes U^*$
- The pairs are subjected to the transformation $U_{XOR^N} \otimes U_{XOR^N}$
- The target pair is measured in the basis $|i\rangle \otimes |j\rangle$
- If the outcomes are equal, the source pair is kept, otherwise it is discarded.

Recurrence protocol for Entanglement Distillation (Illustration)

$$|\psi_+\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle_A |k\rangle_B \,, \; P_+ = |\psi_+\rangle \langle\psi_+|$$

Source: $A_S B_S$; Target: $A_T B_T$

$$|\psi_+\rangle_{A_S B_S} \otimes |\psi_+\rangle_{A_T B_T} = \frac{1}{\sqrt{N}} |i\rangle_{A_S} |j\rangle_{A_T} |i\rangle_{B_S} |j\rangle_{B_T}$$

Alice: $U^{A_S A_T}_{XOR^N} |i\rangle_{A_S} |j\rangle_{A_T} = |i\rangle_{A_S} |i \oplus j\rangle_{A_T}$
Bob: $U^{B_S B_T}_{XOR^N} |i\rangle_{B_S} |j\rangle_{B_T} = |i\rangle_{B_S} |i \oplus j\rangle_{B_T}$
Apply Bilateral XOR:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i,j} |i\rangle_{A_S} |i \oplus j\rangle_{A_T} |i\rangle_{B_S} |i \oplus j\rangle_{B_T}$$

Recurrence protocol for Entanglement Distillation (Illustration)

Measure in the computational basis $|m\rangle$: $P_m = \mathbb{I}_{A_S B_S} \otimes |m\rangle \langle m|_{A_T} \otimes |m\rangle \langle m|_{B_T}$

$$P_m |\psi\rangle = \frac{1}{N} \sum_{i=0}^{N-1} |i\rangle_{A_S} |i\rangle_{B_S} |m\rangle_{A_T} |m\rangle_{B_T}$$

$$\rho^{(m)}_{A_S B_S} \propto Tr_{A_T B_T}(|P_m \psi\rangle \langle P_m \psi|) = \frac{1}{N^2} \sum |ii\rangle \langle i'i'|$$

Recall that $P_+ = |\psi_+\rangle \langle \psi_+|$ with $\psi_+ = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle$.

Up to normalization, $\rho^{(m)}_{A_S B_S}$ is the projector onto $|\psi_+\rangle$.

P_+ sector stays invariant under the recurrence step.

Recurrence protocol in Distillation

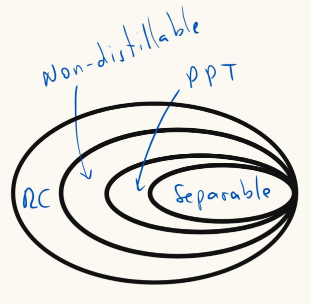
For identical outcomes, do twirling operator to the resulting source pair (note that general twirling procedure leaves P_+ unchanged). The nontrivial change $\to \alpha'$:

$$\alpha'(\alpha) = \alpha \frac{(N(N+1)-2)\alpha + 2}{(N+1)(1+(N-1)\alpha^2)}$$

$\alpha'(\alpha) > \alpha$ for $\alpha > \frac{1}{N+1}$

Conclusion

- Initial question: Is there any NPT state that is non-distillable?
- Corollary from Violating RC $\implies$ distillability:
  - The question reduces to find NPT states that satisfy RC and are non-distillable



- Can be seen that for $N \geq 3$ entangled Werner states satisfy RC while they are NPT
- Therefore, the question amounts to finding an NPT Werner state that is non-distillable
- This is currently an open problem

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- Horodecki, M., Horodecki, P., and Horodecki, R. Mixed-state entanglement and distillation: Is there a “bound” entanglement in nature? 1998.

Questions?

Thank you