Math 595: Representation-theoretic methods in QIT (2022 fall term)

Exercise sheet 2

Last update: September 5, 2022

1. Let $V = \mathbb{C}^d$. Determine bases and the dimensions of the symmetric subspace $\mathrm{Sym}^2(V)$ and the antisymmetric subspace $\mathrm{Alt}^2(V)$.

2. Let (φ, V) be a representation of a group G, and recall that the dual representation (φ^*, V^*) is defined for $g \in G$ and $L \in V^* = \{f \colon V \to \mathbb{C} \text{ linear}\}$ as $\varphi^*(g)(L) := L \circ \varphi(g^{-1})$. Show that:

   (a) $\varphi^*(g) = \varphi(g^{-1})^T$

   (b) $(\varphi^*(g)\langle w|)(\varphi(g)|v\rangle) = \langle w|v\rangle$ for all $\langle w| \in V^*$ and $|v\rangle \in V$.

   (c) (φ^*, V^*) is irreducible if and only if (φ, V) is.

3. Let (φ, V) and (ψ, W) be representations of a group G. Show that the map sending $f \in \hom(V, W)$ to $\psi(g) \circ f \circ \varphi(g^{-1})$ defines a representation of G on $\hom(V, W) = \{f \colon V \to W \text{ linear}\}$.

4. Let (φ, V) and (ψ, W) be representations of a group G, and consider the representation of G on $\hom(V, W)$ defined in Ex. 3 above. Show that $\hom(V, W) \cong V^* \otimes W$ as vector spaces and representations.

5. Let V and W be representations of a group G with characters χ_V and χ_W, respectively. Show that $\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \chi_W$.

6. Denote by $\mathcal{R}(G)$ the regular representation of a finite group G. Show that its character $\chi_{\mathcal{R}(G)}$ satisfies

$$\chi_{\mathcal{R}(G)}(g) = |G| \delta_{e,g}$$

   for $g \in G$, where e denotes the neutral element in G.

7. Show that the multiplicity of any irreducible representation in the regular representation equals its dimension.

8. Let (φ, V) be a representation of a finite group G, where V is a vector space over a field whose characteristic does not divide $|G|$. Show that the operator

$$P = \frac{1}{|G|} \sum_{g \in G} \varphi(g)$$

   is a projection, i.e., $P^2 = P$. What is $\operatorname{im} P$ if $P \neq 0$?