Lecture 38: Determinant of matrices and operators

Last time: Trace of matrices and operators

Determinant: Function $M_n(\mathbb{F}) \to \mathbb{F}$

want: $A \in M_n(\mathbb{F})$ is invertible $\iff \det A \neq 0$

Motivating example: $M_2(\mathbb{F}) \ni A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{F}$

We know (HW): A is invertible $\iff ad - bc \neq 0$

Def Determinant of 2×2-matrices

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{F})$, then the determinant of A, denoted $\det A$, is defined as $\det A = a\cdot d - b\cdot c$.

Properties of the determinant:

- $\det A$ is alternating: it vanishes if two columns are the same.

$$\det \begin{pmatrix} a & a \\ c & c \end{pmatrix} = ac - ac = 0.$$

- $\det A$ is multilinear, i.e., linear in each of its columns:

$$\det \begin{pmatrix} a_1 + \lambda a_2 & b \\ c_1 + \lambda c_2 & d \end{pmatrix} = \det \begin{pmatrix} a_1 & b \\ c_1 & d \end{pmatrix} + \lambda \det \begin{pmatrix} a_2 & b \\ c_2 & d \end{pmatrix}$$

$$\det \begin{pmatrix} a & b_1 + \lambda b_2 \\ c & d_1 + \lambda d_2 \end{pmatrix} = \det \begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix} + \lambda \det \begin{pmatrix} a & b_2 \\ c & d_2 \end{pmatrix}$$

·) det A is normalized: $\det I_2 = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$.

Def Determinant on $M_n(\mathbb{F})$

A determinant is a function $D_n : M_n(\mathbb{F}) \to \mathbb{F}$ that is

·) alternating ($\hat{=}$ vanish if two col's are equal)

·) multilinear ($\hat{=}$ linear in each column)

·) normalized ($\hat{=}$ $D_n(I_n) = 1$)

Prop Existence and uniqueness of determinant

For each $n \in \mathbb{N}$ there is exactly one determinant function $M_n(\mathbb{F}) \to \mathbb{F}$.

(Proof in next lecture)

Prop Let $A \in M_n(\mathbb{F})$ be non-invertible (singular), then $\det(A) = 0$ for any determinant function.

__Proof:__ Recall: $A \in M_n(\mathbb{F})$ is invertible $\iff$ col's of A are lin. indep. as vectors in $\mathbb{F}^n$.

Hence, if A is not invertible, one of the columns of $A = (a_1 | \ldots | a_n)$ is a linear combination of the other ones, say,

$$a_j = \sum_{i \neq j} \lambda_i a_i \,, \quad \lambda_i \in \mathbb{F}, \quad \text{for some } j \in \{1, \ldots, n\}.$$

$$\Rightarrow \det A = \det(a_1|\ldots|a_j|\ldots|a_n)$$

$$= \det\Big(a_1|\ldots|\sum_{i\neq j}\lambda_i a_i|\ldots|a_n\Big)$$

$$= \sum_{i\neq j}\lambda_i \underbrace{\det(a_1|\ldots|a_{j-1}|a_i|a_{j+1}|\ldots|a_n)}_{=0 \text{ by alt. property}} \quad \text{(multilin.)}$$

$$= 0. \qquad \square$$

$\boxed{\text{Prop}}$ Let $D: M_n(\mathbb{F}) \to \mathbb{F}$ be alternating and multilinear, then $D(A) = D(I_n)\det(A)$ for all $A \in M_n(\mathbb{F})$, where det is the unique determinant function on $M_n(\mathbb{F})$.

<u>Proof:</u> Case 1: $D(I_n) \neq 0$.

Define $f: M_n(\mathbb{F}) \to \mathbb{F}$, $f(A) = \dfrac{1}{D(I_n)} D(A)$.

Then clearly f is <u>alternating</u> and <u>multilinear</u> as well (because D is), and $f(I_n) = \dfrac{1}{D(I_n)} D(I_n) = 1$.

By the uniqueness of the determinant, $f(A) = \det(A)\ \forall A \in M_n(\mathbb{F})$.

$$\Rightarrow \det(A)\cdot D(I_n) = D(A) \quad \forall A.$$

Case 2: $D(I_n) = 0$.

We show now that $D(A) = 0 \ \forall A \in M_n(\mathbb{F})$.

Claim: If $A = (a_1 | \dots | a_i | \dots | a_j | \dots | a_n)$,

then $D(A) = -D(A')$ with $A' = (a_1 | \dots | a_j | \dots | a_i | \dots | a_n)$.

Proof: $0 = D(a_1 | \dots | a_i + a_j | \dots | a_j + a_i | \dots | a_n)$

$$= D(a_1 | \dots | a_i | \dots | a_j | \dots | a_n) + D(a_1 | \dots | a_j | \dots | a_i | \dots | a_n)$$

$$+ \underbrace{D(a_1 | \dots | a_i | \dots | a_i | \dots | a_n)}_{=0} + \underbrace{D(a_1 | \dots | a_j | \dots | a_j | \dots | a_n)}_{=0}$$

Now, write every column of A as a linear combination of the standard basis vectors $\{e_1, \dots, e_n\} \subseteq \mathbb{F}^n$.

By multilinearity of D, we can write $D(A)$ as a linear combination of terms $D(e_{i_1} | \dots | e_{i_n})$, $\{e_{i_1}, \dots, e_{i_n}\} = \{e_1, \dots, e_n\}$

But each $D(e_{i_1} | \dots | e_{i_n}) = \pm D(e_1 | \dots | e_n)$

$$= \pm D(I_n)$$

$\Rightarrow D(A) = 0 \ \forall A \in M_n(\mathbb{F})$ if $D(I_n) = 0$

$\Rightarrow D(A) = D(I_n) \det A$. $\square$

$\boxed{\text{Prop}}$ $\det(AB) = \det(A)\det(B)$ for all $A, B \in M_n(\mathbb{F})$.

Proof: Define $D_A(B) = \det(AB)$ for fixed $A \in M_n(\mathbb{F})$.

For $B = (b_1 | \dots | b_n)$, we have $AB = (Ab_1 | \dots | Ab_n)$

$\Rightarrow D_A(B)$ is again alternating and multilinear.

$$\overset{\text{previous}}{\underset{\text{Prop}}{\Longrightarrow}} \quad \det(AB) = D_A(B) = \underline{D_A(I_n)} \cdot \det(B)$$

and $D_A(I_n) = \det(A \cdot I_n) = \det(A)$

$\Rightarrow \det(AB) = \det(A)\det(B)$. $\square$

$\boxed{\text{Cor}}$ $A \in M_n(\mathbb{F})$ is invertible if and only if $\det(A) \neq 0$.

Proof: We know already that $\det A = 0$ if A is singular.

If A is invertible, then $A \cdot A^{-1} = I_n$, and

$$1 = \det(I_n) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1})$$

$\Rightarrow \det(A) \neq 0$. $\square$

$\boxed{\text{Cor}}$ If $A \in M_n(\mathbb{F})$ is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)} = \det(A)^{-1}$.

Cor Let $A, B \in M_n(\mathbb{F})$ be similar matrices. Then $\det(A) = \det(B)$.

Proof: A, B similar: there is $S \in M_n(\mathbb{F})$ invertible with

$$A = SBS^{-1}$$

$$\Rightarrow \det(A) = \det(S \cdot B \cdot S^{-1}) = \det(S) \det(B) \underbrace{\det(S^{-1})}_{= \frac{1}{\det(S)}}$$

$$= \det(B). \qquad \square$$

Def Determinant of an operator

Let $T \in \mathcal{L}_{\mathbb{F}}(V)$. Then the determinant $\det T$ is the determinant of any of its matrix representations, and uniquely defined by the corollary above.

Prop Let $A \in M_n(\mathbb{F})$ be upper-triangular, i.e.,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix}, \quad A_{ij} \in \mathbb{F} \text{ for } 1 \le i \le j \le n.$$

Then $\det(A) = A_{11} \cdot A_{22} \cdot \ldots \cdot A_{nn}$

Proof: With $A = (a_1 | \dots | a_n)$, we have

$$a_1 = A_{11} e_1, \quad a_2 = A_{12} e_1 + A_{22} e_2,$$

$$\dots, \quad a_j = \sum_{i=1}^{j} A_{ij} e_i, \dots, \quad a_n = \sum_{i=1}^{n} A_{in} e_i$$

$$\begin{aligned}
\det(A) &= \det(a_1 | \dots | a_n) \\
&= A_{11} \det(e_1 | a_2 | \dots | a_n) \\
&= A_{11} \Big(A_{12} \underbrace{\det(e_1 | e_1 | a_3 | \dots | a_n)}_{= 0 \text{ by alternating prop.}} + A_{22} \det(e_1 | e_2 | a_3 | \dots | a_n) \Big) \\
&= A_{11} A_{22} \det(e_1 | e_2 | a_3 | \dots | a_n) \\
&\vdots \\
&= A_{11} A_{22} \cdot \ldots \cdot A_{nn} \det \underbrace{(e_1 | e_2 | \dots | e_n)}_{I_n} = A_{11} \cdot \ldots \cdot A_{nn}.
\end{aligned}$$

$\square$

$\boxed{\text{Cor}}$ Let $T \in \mathcal{L}_{\mathbb{C}}(V)$. Then $\det T$ is equal to the product of the eigenvalues of T.

Proof: Use Schur's thm to write T in upper-triangular matrix form (with the eigenvalues on the main diagonal). $\square$