Lecture 29: Self-adjoint and normal operators

Last time: adjoint map

$T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$ is defined via

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle \quad \forall v \in V, w \in W$$

Now $V = W$:

Def 7.11 Self-adjoint operators

An operator $T \in \mathcal{L}_{\mathbb{C}}(V)$ is called self-adjoint, if $T = T^*$.

In other words, $\langle T(v), w \rangle = \langle v, T(w) \rangle \quad \forall v, w \in V$.

Likewise, we call a matrix $A \in M_n(\mathbb{C})$ self-adjoint (Hermitian)

if $A = A^* := (\overline{A})^T$

entry-wise complex conjugate

<u>Ex.</u>: $T: \mathbb{C}^2 \to \mathbb{C}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3x + (2+i)y \\ (2-i)x - 7y \end{pmatrix}$ is self-adjoint:

$v = \begin{pmatrix} x \\ y \end{pmatrix}$, $w = \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{C}^2$ (standard inner product)

$$\langle T(v), w \rangle = \left\langle \begin{pmatrix} 3x + (2+i)y \\ (2-i)x - 7y \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle$$

$$= (3x + (2+i)y)\overline{p} + ((2-i)x - 7y)\overline{q}$$

$$= x\,\overline{(3p + (2+i)q)} + y\,\overline{((2-i)p - 7q)}$$

$$= \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 3p + (2+i)q \\ (2-i)p - 7q \end{pmatrix} \right\rangle = \langle v, T(w) \rangle \Rightarrow T = T^*.$$

matrices (in standard basis): $\mathcal{M}(T) = \begin{pmatrix} 3 & 2+i \\ 2-i & -7 \end{pmatrix}$

$$\mathcal{M}(T^*) = \mathcal{M}(T)^* = \begin{pmatrix} 3 & 2+i \\ 2-i & -7 \end{pmatrix}^* = \begin{pmatrix} 3 & \overline{2-i} \\ \overline{2+i} & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2+i \\ 2-i & -7 \end{pmatrix} = \mathcal{M}(T)$$

Prop 7.6 : if S, T are self-adjoint, then

·) $S+T$ is self-adjoint

·) λS is self-adjoint for $\underline{\lambda \in \mathbb{R}}$

=> self-adjoint operators form a vector space over $\mathbb{R}$.

$\boxed{\text{Prop 7.13}}$ Every eigenvalue of a self-adjoint operator (over $\mathbb{C}$) is real.

<u>Proof</u>: Let $\lambda \in \mathbb{C}$ be an eigenvalue of T with eigenvector $v \neq 0$.

then $\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle T(v), v \rangle \overset{T=T^*}{=} \langle v, T(v) \rangle$

$$= \langle v, \lambda v \rangle = \overline{\lambda} \langle v, v \rangle$$

$\overset{\langle v,v \rangle \neq 0}{\Rightarrow}$ $\lambda = \overline{\lambda}$ or $\lambda \in \mathbb{R}$. $\square$

Prop 7.14 Let V be an inner product space over $\mathbb{C}$, and let $T \in \mathcal{L}_{\mathbb{C}}(V)$. Then $\langle v, T(v)\rangle = 0 \;\forall v \in V$ iff $T = 0$.

Proof: For arbitrary $v, w \in V$, we can rewrite $\langle T(v), w\rangle$ as

$$(*) \quad \langle T(v), w\rangle = \frac{1}{4}\Big(\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle + i\langle T(u+iw), u+iw\rangle - i\langle T(u-iw), u-iw\rangle\Big)$$

(expand and check)

$\Rightarrow$ by assumption ($\langle x, T(x)\rangle = 0 \;\forall x \in V$) and $(*)$,

$$\langle T(v), w\rangle = 0 \quad \forall v, w \in V.$$

for $w = T(v)$: $\langle T(v), T(v)\rangle = 0 \;\forall v \in V \Rightarrow T(v) = 0 \;\forall v \in V$

$$\Rightarrow T = 0$$

$\Leftarrow$ $\langle v, T(v)\rangle = \langle v, 0\rangle = 0 \quad \forall v \in V.$ $\square$

The above Proposition does not hold for operators over $\mathbb{R}$!

(example in class)

$\boxed{\text{Prop 7.15}}$ Let $T \in \mathcal{L}_{\mathbb{C}}(V)$.

$$T \text{ is self-adjoint} \iff \langle T(v), v \rangle \in \mathbb{R} \quad \forall v \in V.$$

Proof: Let $v \in V$ be arbitrary:

$$\begin{aligned}\langle T(v), v \rangle - \overline{\langle T(v), v \rangle} &= \langle T(v), v \rangle - \langle v, T(v) \rangle \\ &= \langle T(v), v \rangle - \langle T^*(v), v \rangle \\ &= \langle (T - T^*)(v), v \rangle \qquad (*)\end{aligned}$$

($\Leftarrow$) if $\langle T(v), v \rangle \in \mathbb{R}$, then $\langle T(v), v \rangle = \overline{\langle T(v), v \rangle}$,

and from $(*)$, $0 = \langle (T - T^*)(v), v \rangle \quad \forall v \in V$

$\Rightarrow T - T^* = 0$ by Prop 7.14 $\Rightarrow T = T^*$.

($\Rightarrow$) If $T = T^*$, then the RHS of $(*)$ is 0, and hence

$\langle T(v), v \rangle = \overline{\langle T(v), v \rangle}$, i.e., $\langle T(v), v \rangle \in \mathbb{R} \ \forall v$.

$\square$

A self-adjoint operator $T \in \mathcal{L}_{\mathbb{R}}(V)$ over a real vector space has a matrix representation given by a symmetric matrix.

A matrix $A \in M_n(\mathbb{R})$ is called symmetric, if $A = A^T$.

The following version of Prop 7.14 holds over $\mathbb{R}$:

Prop 7.16 Let $T \in \mathcal{L}_{\mathbb{R}}(V)$ be self-adjoint.

Then $\langle T(v), v \rangle = 0 \ \forall v \in V \iff T = 0.$

Proof: $\Rightarrow$ Using $\langle T(w), u \rangle \underset{\substack{\uparrow \\ \text{self-adjointness}}}{=} \langle w, T(u) \rangle \underset{\substack{\uparrow \\ \text{symmetry over } \mathbb{R}}}{=} \langle T(u), w \rangle$,

we have $\langle T(u), w \rangle = \frac{1}{4} \left(\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle \right)$

$\Rightarrow$ by assumption $\left(\langle T(x), x \rangle = 0 \ \forall x \in V \right)$, $\langle T(u), w \rangle = 0$

$\forall u, w \in V \quad \Rightarrow \quad T(u) = 0 \ \forall u \in V \Rightarrow T = 0.$

$\Leftarrow$ clear. $\square$

__Normal operators__

Def 7.18 An operator $T \in \mathcal{L}(V)$ is called normal, if it

commutes with its adjoint: $TT^* = T^*T$.

A matrix $A \in M_n(\mathbb{F})$ is called normal, if it commutes with
$(\mathbb{F} = \mathbb{R}, \mathbb{C})$
its conjugate transpose: $AA^* = A^*A$.

Clear: If T is self-adjoint, then T is normal:

$$TT^* = T^2 = T^*T$$

Ex.: $A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$, then $A^* \neq A$, but $A^*A = AA^*$:

$$AA^* = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} = A^*A.$$

$\boxed{\text{Prop 7.20}}$ $T \in \mathcal{L}(V)$ is normal

$$\iff \|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle T^*(v), T^*(v)\rangle = \|T^*(v)\|^2 \quad \forall v \in V.$$

Proof: T is normal $\iff TT^* - T^*T = 0$

$$\iff \langle (TT^* - T^*T)(v), v\rangle = 0 \quad \forall v \in V$$

$TT^* - T^*T$ is self-adjoint, so this step holds over both $\mathbb{R}$ and $\mathbb{C}$!

$$\iff \langle TT^*(v), v\rangle = \langle T^*T(v), v\rangle \quad \forall v \in V$$

$$\iff \langle T^*(v), T^*(v)\rangle = \langle T(v), T(v)\rangle \quad \forall v \in V.$$

$\square$