Lecture 27: Orthogonal projections

Last time: Orthogonal complements

V finite-dim. inner product space, $U \leq V$ subspace: $V = U \oplus U^\perp$

$\boxed{\text{Def } 6.53}$ Orthogonal projection

Let V be a finite-dim. vector space, $U \leq V$ a subspace.

The orthogonal projection of V onto U, denoted P_U, is an operator in $\mathcal{L}(V)$ defined as follows: For $v \in V$, let $u \in U$ and $w \in U^\perp$ be the unique vectors s.t. $v = u + w$. Then $P_U(v) = u \in U$.

$\boxed{\text{Prop } 6.55}$ Properties of the orthogonal projection

V finite-dim. VS, $U \leq V$ a subspace, P_U the map as defined in Def. 6.53.

i) $P_U \in \mathcal{L}(V)$ (i.e., P_U is linear)

ii) $P_U(u) = u \quad \forall u \in U$

iii) $P_U(w) = 0 \quad \forall w \in U^\perp$

iv) $\operatorname{im} P_U = U$

v) $\ker P_U = U^\perp$

vi) $v - P_U(v) \in U^\perp \quad \forall v \in V$

vii) $P_U^2 = P_U$

viii) $\|P_U(v)\| \leq \|v\|$

$(\|v\| = \sqrt{\langle v, v \rangle})$

ix) Let $\{u_1, \dots, u_m\}$ be an ONB for U, then $P_U(v) = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_m \rangle u_m$

Proof: i) Let $v_1, v_2 \in V$, $v_1 = u_1 + w_1$ with $u_i \in U$, $w_i \in U^\perp$.
$v_2 = u_2 + w_2$

Then
$$P_U(v_1+v_2) = P_U(u_1+w_1+u_2+w_2)$$
$$= P_U(\underbrace{u_1+u_2}_{\in U} + \underbrace{w_1+w_2}_{\in U^\perp})$$
$$= u_1 + u_2$$
$$= P_U(v_1) + P_U(v_2)$$

$P_U(\lambda v) = \lambda P_U(v)$ for $\lambda \in \mathbb{C}$, $v \in V$ is proved similarly.

ii) If $u \in U$, then $\underset{\in U}{u} = u + \underset{\in U^\perp}{0}$, so $P_U(u) = u$.

iii) If $w \in U^\perp$, then $w = \underset{\in U}{0} + \underset{\in U^\perp}{w}$, so $P_U(w) = 0$.

iv) $U \subseteq \operatorname{im} P_U$ by (ii), and $\operatorname{im} P_U \subseteq U$ by def. $\Rightarrow \operatorname{im} P_U = U$.

v) $U^\perp \subseteq \ker P_U$ by (iii). If $v \in \ker P_U$, then
$$v = 0 + v \text{ with } v \in U^\perp \quad (P_U(v) = 0)$$
so $\ker P_U \subseteq U^\perp \Rightarrow \ker P_U = U^\perp$.

vi) Let $v \in V$, $v = u + w$, $u \in U$, $w \in U^\perp$: $v - \overbrace{P_U(v)}^{=u} = u + w - u = \underset{\in U^\perp}{w}$.

vii) $P_U^2 = P_U$: Let $v \in V$, $v = u + w$, $u \in U$, $w \in U^\perp$.

$$P_U^2(v) = P_U(P_U(v)) = P_U(P_U(u+w))$$
$$= P_U(u)$$
$$= u$$
$$= P_U(v) \quad \Rightarrow \quad P_U^2 = P_U.$$

viii) $\|P_U(v)\|^2 = \|u\|^2 \leq \|u\|^2 + \|w\|^2 = \|v\|^2$

$$\langle v, v\rangle = \langle u+w, u+w\rangle$$
$$= \langle u,u\rangle + \langle w,w\rangle + \underbrace{\langle u,w\rangle}_{=0} + \underbrace{\langle w,u\rangle}_{=0}$$

ix) $u \in U$: $u = \lambda_1 u_1 + \ldots + \lambda_m u_m$ for some $\lambda_i \in \mathbb{C}$.

Let $v = u + w$, $w \in U^\perp \Rightarrow P_U(v) = u = \lambda_1 u_1 + \ldots + \lambda_m u_m$

$$\langle v, u_j\rangle = \langle u + w, u_j\rangle = \underbrace{\langle u, u_j\rangle}_{=\lambda_j} + \underbrace{\langle w, u_j\rangle}_{=0} = \lambda_j.$$

$\square$

$\boxed{\text{Prop}}$ Let $U \leq V$ be a subspace and P_U the orthogonal projection onto U. Let $B = \{u_1, \dots, u_m, w_1, \dots, w_k\}$ be a basis for V, where $\{u_1, \dots, u_m\}$ and $\{w_1, \dots, w_k\}$ form a basis for U and $U^\perp$, respectively. Then,

$$\mathcal{M}(P_U)_{B,B} = \begin{array}{c} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}0\\ \ddots\\ 0\end{matrix}}\right\} k \end{array} \\ \underbrace{\hphantom{1 \ \ddots \ 1}}_{m}\underbrace{\hphantom{0\ \ddots\ 0}}_{k} \end{array} \qquad (*)$$

That is, P_U is diagonalizable.

Proof: ·) If $U = \{0\}$, then $m = 0$, $U^\perp = V$ ($k = \dim V$), and

$$\mathcal{M}(P_U)_{B,B} = 0.$$

·) If $U = V$, then $m = \dim V$, $U^\perp = \{0\}$ ($k = 0$), and

$$\mathcal{M}(P_U)_{B,B} = I_{\dim V}.$$

·) Assume $U \leq V$, $U \neq \{0\}, V$. Then by Prop 6.55,

$\rightarrow$ $P_U(u) = u$ for all $u \in U$ $\Rightarrow$ $\lambda_1 = 1$ is an eigenvalue, and $E(1, P_U) = U$.

$\Rightarrow P_U(w) = 0 \quad w \in U^\perp \Rightarrow \lambda_2 = 0$ is an eigenvalue,

$$Eig(0, P_U) = U^\perp$$

$\Rightarrow V = U \oplus U^\perp = Eig(\lambda_1, P_U) \oplus Eig(\lambda_2, P_U)$

$\Rightarrow P_U$ is diagonalizable by Prop 5.41

Choosing bases $\{u_1, \dots, u_m\}$ for U, $\{w_1, \dots, w_k\}$ for $U^\perp$,

we have $P_U(u_i) = u_i \quad \forall i = 1, \dots, m$, and $P_U(w_j) = 0 \quad \forall j = 1, \dots, k$

$\Rightarrow$ matrix representation in $(*)$. $\square$

Application: minimization problems

Given subspace $U \leq V$ and $v \in V$.

Goal: minimize $\|v - u\|$ over all $u \in U$ (find closest vector to v in U)

$\boxed{\text{Prop 6.56}}$ Let V be finite-dim., $U \leq V$ subspace, and fix $v \in V$.

For all $u \in U$, $\|v - P_U(v)\| \leq \|v - u\|$

We have equality iff $u = P_U(v)$.

Proof: $\|v-u\|^2 = \|\underbrace{v - P_U(v)}_{=: w \in U^\perp} + \underbrace{P_U(v) - u}_{=: \tilde{u} \in U}\|^2$

$$= \langle w + \tilde{u}, w + \tilde{u} \rangle$$

$$= \langle w, w \rangle + \langle \tilde{u}, \tilde{u} \rangle + \underbrace{\langle w, \tilde{u} \rangle}_{=0} + \underbrace{\langle \tilde{u}, w \rangle}_{=0}$$

$$= \|w\|^2 + \|\tilde{u}\|^2$$

$$= \|v - P_U(v)\|^2 + \underbrace{\|P_U(v) - u\|^2}_{\geq 0}$$

$$\geq \|v - P_U(v)\|^2 .$$

Equality iff $\|P_U(v) - u\|^2 = 0$ iff $P_U(v) = u$. $\square$