Lecture 21: Existence of eigenvalues, upper-triangular matrices

Last time: Invariant subspaces, eigenvalues, eigenvectors

Eigenvalue equation: Let $T \in \mathcal{L}_{\mathbb{F}}(V)$, then $v \in V$, $v \neq 0$ is called an eigenvector with eigenvalue $\lambda \in \mathbb{F}$, if $T(v) = \lambda v$.

Prop 5.6 Let V be a finite-dim. VS, $T \in \mathcal{L}_{\mathbb{F}}(V)$, and $\lambda \in \mathbb{F}$.

TFAE: i) λ is an eigenvalue of T.

ii) $T - \lambda I_V$ is not injective.

iii) $T - \lambda I_V$ is not surjective.

iv) $T - \lambda I_V$ is not invertible.

__Proof:__ λ is an eigenvalue of $T \iff \exists v \neq 0$ with $T(v) = \lambda v$

$$\iff \exists v \neq 0 : (T - \lambda I_V)(v) = 0$$

$$\iff \ker(T - \lambda I_V) \neq \{0\}$$

$$\iff T - \lambda I_V \text{ is not injective.}$$

$\Rightarrow$ (i) $\iff$ ii))

ii) $\iff$ iii) $\iff$ iv) follows from Lecture 17. $\square$

Existence of eigenvalues

Remarks: ·) Let $p \in P_d(\mathbb{F})$ be a polynomial over $\mathbb{F}$ with degree at most d, $p = a_0 + a_1 x + \ldots + a_d x^d$, $a_i \in \mathbb{F}$, and let $T \in \mathcal{L}_{\mathbb{F}}(V)$.

Then
$$p(T) = a_0 \underset{\substack{\parallel\!\parallel \\ T^0}}{I_V} + a_1 T + \ldots + a_d \underset{\substack{\parallel\!\parallel \\ \underbrace{T \circ \ldots \circ T}_{d \text{ times}}}}{T^d} \in \mathcal{L}_{\mathbb{F}}(V).$$

·) Fundamental theorem of algebra: $(p = a_0 + a_1 z + \ldots + a_d z^d)$

Let $p \in P_d(\mathbb{C})$ be a polynomial over $\mathbb{C}$ of degree d (i.e., $a_d \neq 0$).

Then up to the order of factors, p can be uniquely written as

$$p = c\,(z - \lambda_1) \ldots (z - \lambda_d)$$

with $c, \lambda_i \in \mathbb{C}$. The λ_i are the roots of p, i.e., $p(\lambda_i) = 0$

$\forall i = 1, \ldots, d$.

$\boxed{\text{Prop 5.21}}$ Operators on complex vector spaces have eigenvalues

Let V be a complex VS, $V \neq \{0\}$, and let $T \in \mathcal{L}_{\mathbb{C}}(V)$, then there exists $v \in V$, $v \neq 0$ and $\lambda \in \mathbb{C}$ s.t. $T(v) = \lambda v$.

Proof: Let $\dim V = n > 0$, and let $v \in V$, $v \neq 0$ be arbitrary.

Consider the following $n+1$ vectors: $v, T(v), T^2(v), \dots, T^n(v)$

Then $\{v, T(v), \dots, T^n(v)\}$ are linearly dependent (since $\dim V = n$),

and so there are $a_i \in \mathbb{C}$, $i = 0, \dots, n$, not all of them $= 0$,

$$0 = a_0 v + a_1 T(v) + \dots + a_n T^n(v) \qquad (*)$$

In fact, $a_i \neq 0$ for some $i \geq 1$ (since otherwise $v = 0$ ↯).

Consider then the non-constant polynomial

$$p = a_0 + a_1 z + \dots + a_n z^n$$

$$= c\,(z - \lambda_1) \dots (z - \lambda_m) \quad \text{for } m \leq n \text{ and } c, \lambda_i \in \mathbb{C}.$$

by the fundamental theorem of algebra.

Now, $(*)$ becomes
$$\begin{aligned} 0 &= a_0 + a_1 T(v) + \dots + a_n T^n(v) \\ &= (a_0 + a_1 T + \dots + a_n T^n)(v) \\ &= c\,(T - \lambda_1 I_V) \cdot (T - \lambda_2 I_V) \dots (T - \lambda_m I_V)(v) \end{aligned}$$

Since $v \neq 0$, $(T - \lambda_i I_V)(w) = 0$ for some $i = 1, \dots, m$ and non-zero $w \in V$,

and hence $T - \lambda_i I_V$ is not injective and

$\lambda_i \in \mathbb{C}$ is an eigenvalue of T by Prop 5.6. $\square$

Upper-triangular matrices

$\boxed{\text{Def 5.25}}$ A matrix $A \in M_n(\mathbb{F})$ is called upper-triangular, if $A_{ij} = 0$ whenever $i > j$, i.e.,

$$A = \begin{pmatrix} * & & & * \\ 0 & \ddots & & \\ \vdots & 0 & \ddots & \\ 0 & \cdots & 0 & * \end{pmatrix}.$$

$\boxed{\text{Prop 5.26}}$ Let $T \in \mathcal{L}_{\mathbb{F}}(V)$ and $\mathcal{B}_V = \{v_1, \dots, v_n\}$ be a basis for V. Set $A = \mathcal{M}(T)_{\mathcal{B}_V, \mathcal{B}_V}$. TFAE:

i) A is upper-triangular

ii) $T(v_j) \in \langle v_1, \dots, v_j \rangle$ for each $j = 1, \dots, n$.

iii) $U_j = \langle v_1, \dots, v_j \rangle$ is invariant under T for each $j = 1, \dots, n$.
$(T(U_j) \subseteq U_j)$

<u>Proof:</u> i) $\Leftrightarrow$ ii): Recall def. of $A = \mathcal{M}(T)_{\mathcal{B}_V, \mathcal{B}_V}$

$$T(v_j) = \sum_{i=1}^{n} A_{ij} v_i$$

Hence, A is upper-triangular $\Leftrightarrow A_{ij} = 0$ for $i > j$

$$\Leftrightarrow T(v_j) = \sum_{i=1}^{j} A_{ij} v_j$$

$$\Leftrightarrow T(v_j) \in \langle v_1, \dots, v_j \rangle$$

iii) => ii) if $U_j = \langle v_1, \dots, v_j \rangle$ is invariant under T,

then $T(v_j) \in U_j = \langle v_1, \dots, v_j \rangle$

ii) => iii) Fix $j \in \{1, \dots, n\}$.

By assumption ii):
$$\left.\begin{aligned} T(v_1) &\in \langle v_1 \rangle \subseteq U_j \\ T(v_2) &\in \langle v_1, v_2 \rangle \subseteq U_j \\ &\vdots \\ T(v_j) &\in \langle v_1, \dots, v_j \rangle = U_j \end{aligned}\right\} (*)$$

Let now $v \in \underset{\overset{\parallel}{U_j}}{\langle v_1, \dots, v_j \rangle}$, then $v = \sum_{i=1}^{j} \lambda_i v_i$ for some $\lambda_i \in \mathbb{F}$,

and $T(v) = \sum_{i=1}^{j} \lambda_i \underbrace{T(v_i)}_{\in U_j \text{ by } (*)} \in U_j$ => U_j is invariant under T => iii) $\square$

$\boxed{\text{Prop 5.27}}$ Let V be a finite-dim. VS over $\mathbb{C}$, and $T \in \mathcal{L}_{\mathbb{C}}(V)$. Then there exists a basis B_V for V s.t. $\mathcal{M}(T)_{B_V, B_V}$ is upper-triangular.

<u>Proof</u>: We use induction over $\dim V$.

$\dim V = 1$: Nothing to show, because $\mathcal{M}(T)$ is a 1×1-matrix and hence upper-triangular.

$\dim V > 1$. Suppose the result is true for all complex vector spaces of dim. $< \dim V$.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of T (which exists by Prop 5.21), and set $U = \operatorname{im}(T - \lambda I_V)$.

Since $T - \lambda I_V$ is not surjective by Prop 5.6, we have $\dim U < \dim V$, and furthermore U is invariant under T:

let $u \in U$, then $T(u) = T(u) - \lambda u + \lambda u$

$$= \underbrace{(T - \lambda I_V)(u)}_{\in U} + \underbrace{\lambda u}_{\in U} \in U.$$

$\Rightarrow T|_U \in \mathcal{L}_{\mathbb{C}}(U)$ and $\dim U < \dim V$

by the induction hypothesis, there is a basis $B_U = \{u_1, \dots, u_m\}$ s.t. $T|_U$ has an upper-triangular matrix, i.e.,

for $j = 1, \dots, m$: $\quad T(u_j) \in \langle u_1, \dots, u_j \rangle$

Extend B_U to a basis $B_V = \{u_1, \dots, u_m, v_1, \dots, v_n\}$ for V.

Let $k \in \{1, \dots, n\}$: $T(v_k) = T(v_k) - \lambda v_k + \lambda v_k$

$$= \overset{\in U}{(T - \lambda I_V)(v_k)} + \lambda v_k \in \langle u_1, \dots, u_m, v_1, \dots, v_k \rangle$$

$\Rightarrow \mathcal{M}(T)_{B_V, B_V}$ is upper-triangular by Prop. 5.26. $\square$

Prop 5.30 Let $A \in M_n(\mathbb{F})$ be an upper-triangular matrix.

Then A is invertible, if and only if all diagonal elements of A are non-zero, $A = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$, $c_i \neq 0$.

Proof: Use the inversion algorithm for A:

All diagonal elements of A are non-zero iff the RREF of the augmented matrix $(A \mid I_n)$ is of the form $(I_n \mid B)$ $B = A^{-1}$ $\square$

Prop 5.32 Let $T \in \mathcal{L}_{\mathbb{F}}(V)$ and $A = \mathcal{M}(T)_{B_V, B_V}$ is upper-triangular w.r.t. some basis B_V for V.

Then the diagonal elements of A are precisely the eigenvalues of T.

Proof: Let $A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. Then $\mathcal{M}(T - \lambda I_V)$ has the form $\mathcal{M}(T - \lambda I_V) = \begin{pmatrix} \lambda_1 - \lambda & & * \\ & \ddots & \\ 0 & & \lambda_n - \lambda \end{pmatrix}$

By Prop 5.30, $\mathcal{M}(T - \lambda I_V)$ (and hence $T - \lambda I_V$) is not invertible iff $\lambda = \lambda_i$ for some $i = 1, \dots, n$, i.e., λ_i is an eigenvalue of T. $\square$