Lecture 20: Invariant subspaces, eigenvalues, eigenvectors

Last time: Rank of a matrix

$\mathcal{L}_{\mathbb{F}}(V,V) \equiv \mathcal{L}_{\mathbb{F}}(V)$, an element $T \in \mathcal{L}_{\mathbb{F}}(V)$ is typically called an operator.

Goal: study the structure of operators via invariance properties

$\boxed{\text{Def}}$ Let $T \in \mathcal{L}_{\mathbb{F}}(V)$. A subspace $U \leq V$ is called invariant under T, if $T(u) \in U$ for all $u \in U$. $\left(\underset{\{T(u) \,:\, u \in U\}}{\underset{\parallel}{T(U)}} \subseteq U \right)$

Ex.: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x+3y \\ y \end{pmatrix}$

Set $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $T(v) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2v$

$\Rightarrow$ $\langle v \rangle$ is an invariant subspace of T of dim. 1.

Ex.: Let $P: \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x-z \\ y-z \\ 0 \end{pmatrix}$ $\quad (P^2 = P)$

$U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ is an invariant subspace of dim. 2.

The following 4 subspaces are always invariant subspaces of a map $T \in \mathcal{L}_{\mathbb{F}}(V)$:

- V (clear)
- $\ker T$ ($\forall v \in \ker T$, $T(v) = 0 \in \ker T$)
- $\{0\}$ ($T(0) = 0$ by linearity)
- $\operatorname{im} T$ ($T(w) \in \operatorname{im} T$ for all $w \in \operatorname{im} T$)

Restrict our attention to 1-dim. invariant subspaces:

Let $v \in V, v \neq 0$, $U = \langle v \rangle = \{\lambda v : \lambda \in \mathbb{F}\}$

Then U is an invariant subspace for T

$\iff \forall v \in U$, $\underline{T(v) = \lambda v}$ for some $\lambda \in \mathbb{F}$.

Def 5.5, 5.7 Eigenvalues, eigenvectors

Let $T \in \mathcal{L}_{\mathbb{F}}(V)$. If $v \in V$, $\underline{v \neq 0}$ satisfies

$$T(v) = \lambda v \text{ for some } \lambda \in \mathbb{F},$$

then λ is called an eigenvalue of T with eigenvector v $(\neq 0)$.

Ex.: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x+3y \\ y \end{pmatrix}$

then $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ satisfies $T(v) = 2v = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, so v is an eigenvector of T with eigenvalue 2.

Ex.: $P: \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x-z \\ y-z \\ 0 \end{pmatrix}$

Then $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ satisfies $P(v_1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \cdot v_1 \Rightarrow \lambda_1 = 1$

$v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ satisfies $P(v_2) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 \cdot v_2 \Rightarrow \lambda_2 = 1$

(P acts as the identity on $U = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rangle$)

$v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ satisfies $P(v_3) = 0 = 0 \cdot v_3 \Rightarrow \lambda_3 = 0$

Ex.: If $\dim \ker T > 0$, then every $v \in \ker T$, $v \neq 0$, is an eigenvector of T with eigenvalue 0.

$\boxed{\text{Prop 5.10}}$ Let $T \in \mathcal{L}_{\mathbb{F}}(V)$, let $\lambda_1, \dots, \lambda_m \in \mathbb{F}$ be distinct eigenvalues of T (i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$), and $v_1, \dots, v_m \in V$ the corresponding (non-zero) eigenvectors,

$$T(v_i) = \lambda_i v_i \quad \text{for } i = 1, \dots, m.$$

Then $\{v_1, \dots, v_m\}$ are linearly independent.

Proof: By contradiction: assume $\{v_1, \dots, v_m\}$ are linearly dependent.

Let k be the smallest index s.t.

·) $\{v_1, \dots, v_k\}$ are linearly independent

·) $v_{k+1} \in \langle v_1, \dots, v_k \rangle$

Hence, there are $a_i \in \mathbb{F}$ s.t. $v_{k+1} = a_1 v_1 + \ldots + a_k v_k$ $(*)$

apply T to $(*)$ and use $T(v_i) = \lambda_i v_i$:

$$\lambda_{k+1} v_{k+1} = T(a_1 v_1 + \ldots + a_k v_k) = a_1 \lambda_1 v_1 + \ldots + a_k \lambda_k v_k \quad (**)$$

multiply $(*)$ by λ_{k+1} and subtract it from $(**)$:

$$0 = a_1 (\underbrace{\lambda_1 - \lambda_{k+1}}_{\neq 0}) v_1 + \ldots + a_k (\underbrace{\lambda_k - \lambda_{k+1}}_{\neq 0}) v_k$$

$\{v_1, \ldots, v_k\}$ are linearly independent and $\lambda_i - \lambda_{k+1} \neq 0$ for $i = 1, \ldots, k$ $\Rightarrow$ $a_i = 0$ for $i = 1, \ldots, k$.

By $(*)$, $v_{k+1} = a_1 v_1 + \ldots + a_k v_k = 0 v_1 + \ldots + 0 v_k = 0$

$\Rightarrow$ contradiction to $v_{k+1} \neq 0$ $\Rightarrow$ claim. $\square$

$\boxed{\text{Cor } 5.13}$ If V is finite-dim., then every $T \in \mathcal{L}_{\mathbb{F}}(V)$ has at most $\dim V$ distinct eigenvalues.

Proof: the number of linearly independent vectors in V is at most $\dim V$ $\Rightarrow$ claim by previous Prop. $\square$

Ex.: Let $T : \mathbb{F}^2 \to \mathbb{F}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$

Find eigenvalues of T for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$.

Want to find $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}^2$ s.t. $T(v) = \begin{pmatrix} -y \\ x \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda v$

for some $\lambda \in \mathbb{F}$.

$\Rightarrow$ $-y = \lambda x$, $x = \lambda y$ $\Rightarrow$ $x = -\lambda^2 x$ or $x(1+\lambda^2) = 0$

so either $x = 0$ or $1 + \lambda^2 = 0$.

Since eigenvectors are non-zero by def., $x \neq 0$.

$\Rightarrow$ $1 + \lambda^2 = 0$. This equation has no solutions over $\mathbb{R}$

$(\lambda^2 \geq 0$ for all $\lambda \in \mathbb{R})$

$\Rightarrow$ $T : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix}$ has no eigenvectors/eigenvalues

over $\mathbb{C}$, $1 + \lambda^2 = 0$ has the solutions $\lambda = \pm i$ $(i^2 = -1)$

Let $v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, then $T(v_1) = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix} = i v_1$

$v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, then $T(v_2) = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix} = -i v_2$

$\{v_1, v_2\}$ are linearly independent.