Lecture 17: Invertibility and basis change

Last time: Invertibility of linear maps and isomorphic vector spaces

Recall lecture 14: VS's V and W over $\mathbb{F}$, $\dim V = n$, $\dim W = m$,

fix bases B_V, B_W in V and W, resp.

The map $\mathcal{M}(.)_{B_V,B_W} : \mathcal{L}_{\mathbb{F}}(V,W) \to M_{m,n}(\mathbb{F})$

is a (linear) isomorphism. If U,V,W are VS's over $\mathbb{F}$ with

bases B_U, B_V, B_W, then

$$\mathcal{M}(ST)_{B_U,B_W} = \mathcal{M}(S)_{B_V,B_W} \cdot \mathcal{M}(T)_{B_U,B_V}, \qquad (*)$$

where $S \in \mathcal{L}_{\mathbb{F}}(V,W)$, $T \in \mathcal{L}_{\mathbb{F}}(U,V)$.

$\boxed{\text{Prop}}$ Basis change

Let B and B' be two bases for V, $\dim V = n$, then $\mathcal{M}(I_V)_{B,B'}$ $\underset{\cup}{}$ $M_n(\mathbb{F})$

is an invertible matrix with inverse $\mathcal{M}(I_V)_{B,B'}^{-1} = \mathcal{M}(I_V)_{B',B}$.

(Here, $I_V : V \to V$, $v \mapsto v$ is the identity map)

<u>Proof</u>: Clearly, $I_V \circ I_V = I_V^2 = I_V$, and for any basis B of V,

by def. $\mathcal{M}(I_V)_{B,B} = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

From $(*)$, we get $I_n = \mathcal{M}(I_V)_{B,B} = \mathcal{M}(I_V^2)_{B,B} = \mathcal{M}(I_V)_{B',B} \cdot \mathcal{M}(I_V)_{B,B'}$.

$\square$

Prop Let $T \in \mathcal{L}_{\mathbb{F}}(V,W)$ (V,W finite-dim. VS's)

The following are equivalent:

i) T is invertible

ii) $\mathcal{M}(T)_{B_V,B_W}$ is invertible for any bases B_V for V and B_W for W.

__Proof:__ i) $\Rightarrow$ ii) If T is invertible, then $T^{-1} \in \mathcal{L}_{\mathbb{F}}(W,V)$ exists and satisfies $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$.

Now, let B_V and B_W be arbitrary bases for V and W, resp.

then, with $\dim V = n$,
$$\begin{aligned} I_n &= \mathcal{M}(I_V)_{B_V,B_V} \\ &= \mathcal{M}(T^{-1} \circ T)_{B_V,B_V} \\ &\overset{(*)}{=} \mathcal{M}(T^{-1})_{B_W,B_V} \cdot \mathcal{M}(T)_{B_V,B_W} \end{aligned}$$

Hence, $\mathcal{M}(T)_{B_V,B_W}$ is invertible with inverse $\mathcal{M}(T^{-1})_{B_W,B_V}$.

ii) $\Rightarrow$ i) Let $T: V \to W$ be a linear map, and for fixed but arbitrary bases B_V for V and B_W for W, let $A = \mathcal{M}(T)_{B_V,B_W}$ be invertible. We know: $\underset{v \in V}{[T(v)]_{B_W}} = \mathcal{M}(T)_{B_V,B_W} \cdot \mathcal{M}(v)_{B_V}$
$$= A \cdot [v]_{B_V} = [w]_{B_W}$$

We know: $[T(v)]_{B_W} = \mathcal{M}(T)_{B_V, B_W} \cdot \mathcal{M}(v)_{B_V}$
$v \in V$

$$= A \cdot [v]_{B_V} = [w]_{B_W}$$

Define a linear map $S: W \to V$ via $[S(w)]_{B_V} := A^{-1} \cdot [w]_{B_W}$

Then $[S(T(v))]_{B_V} = A^{-1} \cdot [T(v)]_{B_W}$

$$= A^{-1} \cdot (A \cdot [v]_{B_V})$$

$$= \underbrace{A^{-1} \cdot A}_{I_n} \cdot [v]_{B_V} = [v]_{B_V}$$

$\Rightarrow S(T(v)) = v$ for all $v \in V$ $\Rightarrow$ $S = T^{-1}$. $\square$

Let us now w.l.o.g. assume that $W = V$ (since $V \cong W$ if and only if there exists an isomorphism $T: V \to W$, and then $\dim V = \dim W$).

$\boxed{\text{Prop}}$ Let V be a VS over $\mathbb{F}$ with $\dim V = n$. Let $T \in \mathcal{L}_{\mathbb{F}}(V, V)$ and set $A = \mathcal{M}(T)_{B, B'}$ w.r.t. some bases B and B' for V.

TFAE:

i) T is invertible
ii) T is injective
iii) T is surjective
iv) A is invertible
v) The columns of A are lin. independent as vectors in $\mathbb{F}^n$
vi) The columns of A span $\mathbb{F}^n$.

Proof: First, we show i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii)

i) $\Rightarrow$ ii) true by Prop 3.56 (invertible $\Leftrightarrow$ inj. + surj.)

ii) $\Rightarrow$ iii) T is injective $\Rightarrow$ $\ker T = \{0\}$ by Prop 3.16

$$\Rightarrow \dim V = \dim \ker T + \dim \operatorname{im} T \quad \text{(Prop 3.22)}$$

$$= 0 + \dim \operatorname{im} T$$

$\Rightarrow \operatorname{im} T = V$ (midterm exam) $\Rightarrow$ T surjective.

iii) $\Rightarrow$ i) Again, by dimension formula,

$$\dim \ker T = \dim V - \dim \underbrace{\operatorname{im} T}_{=V} = 0$$

$\Rightarrow \ker T = \{0\}$ $\Rightarrow$ T is injective $\Rightarrow$ T invertible.

i) $\Leftrightarrow$ iv) true by previous Prop.

Remains to be shown: iv) $\Leftrightarrow$ v) $\Leftrightarrow$ vi)

iv) $\Rightarrow$ v) Recall: if $B = (b_1 | \dots | b_n)$, then $\underline{AB = (Ab_1 | \dots | Ab_n)}$

$$\underset{\substack{\parallel \\ (a_1 | \dots | a_n)}}{A^{-1} \cdot A} = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & 1 \end{pmatrix} = (e_1 | \dots | e_n) = (A^{-1} a_1 | \dots | A^{-1} a_n)$$

$$\Rightarrow A^{-1} a_i = e_i \quad 1 \le i \le n.$$

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$a_i \in \mathbb{F}^n$ ... columns of the matrix A.

Assume $\lambda_1 a_1 + \ldots + \lambda_n a_n = 0$ for some $\lambda_i \in \mathbb{F}$.

Multiply by A^{-1}: $\lambda_1 \underbrace{A^{-1} a_1}_{e_1} + \ldots + \lambda_n \underbrace{A^{-1} a_n}_{e_n} = A^{-1} 0 = 0$

But $\{e_1, \ldots, e_n\}$ are lin. independent, and hence $\lambda_i = 0 \ \forall i$.

$\Rightarrow \{a_1, \ldots, a_n\}$ are lin. indep.

v) $\Rightarrow$ vi) know: $\dim \mathbb{F}^n = n$, and we have n lin. independent vectors $a_1, \ldots, a_n \Rightarrow \{a_1, \ldots, a_n\}$ is a basis for $\mathbb{F}^n$,

and in particular $\langle a_1, \ldots, a_n \rangle = \mathbb{F}^n$.

vi) $\Rightarrow$ iv) to show: A invertible, i.e., there exists $B \in M_n(\mathbb{F})$ with $A \cdot B = B \cdot A = I_n$.

Recall that $(I_n)_{ij} = \delta_{ij}$ for $1 \le i, j \le n$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{F}^n$, $(e_j)_i = \delta_{ij}$

Let $\{a_1, \ldots, a_n\}$ be the column vectors of A, $(a_k)_i = A_{ik}$

Since $\langle a_1, \ldots, a_n \rangle = \mathbb{F}^n$ by assumption, we can write

$$e_j = \sum_{k=1}^{n} \lambda_{kj} a_k \quad \text{for } 1 \le j \le n.$$

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$$\underset{\substack{\| \\ (I_n)_{ij}}}{(e_j)_i = \delta_{ij}} = \sum_{k=1}^{n} \lambda_{kj} \underbrace{(a_k)_i}_{= A_{ik}} = \sum_{k=1}^{n} A_{ik} \lambda_{kj}$$

$\Rightarrow$ defining $B \in M_n(\mathbb{F})$ via $(B)_{ij} = \lambda_{ij}$, we have

$$I_n = A \cdot B \quad \Rightarrow \quad B = A^{-1} \text{ and } A$$

is invertible. $\square$