Lecture 16: Invertibility of linear maps and isomorphic vector spaces

Last time: Invertibility of matrices

Def 3.53 A linear map $T \in \mathcal{L}_{\mathbb{F}}(V,W)$ is called invertible if there exists another linear map $S \in \mathcal{L}_{\mathbb{F}}(W,V)$ s.t.

$ST = I_V : V \to V$ and $TS = I_W : W \to W$.

S is then called an inverse for T.

Prop 3.54 If $T \in \mathcal{L}_{\mathbb{F}}(V,W)$ is invertible, then its inverse $S \in \mathcal{L}_{\mathbb{F}}(W,V)$ is unique, denoted T^{-1}.

Proof: Same as for matrices: Let S_1, S_2 ($W \to V$) be inverses for T, then

$$S_1 = S_1 \cdot I_W = S_1 (TS_2) = (S_1 T) S_2 = I_V \cdot S_2 = S_2. \qquad \square$$

Prop 3.56 A linear map is invertible if and only if it is injective and surjective ($\hat{=}$ bijective).

Proof: ($\Rightarrow$) Let $T \in \mathcal{L}_{\mathbb{F}}(V,W)$ be invertible with inverse $T^{-1} \in \mathcal{L}_{\mathbb{F}}(W,V)$:

- T is injective: Let $v \in \ker T$, i.e. $T(v) = 0$.

$$\text{Then } \underbrace{T^{-1}(T(v))}_{= v} = T^{-1}(0) = 0$$

$$\Rightarrow v = 0 \Rightarrow \ker T = \{0\}.$$

·) T is surjective: Let $w \in W$ be arbitrary.

Since $T \circ T^{-1} = I_W$, we have $T(T^{-1}(w)) = w$

But $T^{-1}(w) \in V$, so $w \in \operatorname{im} T \Rightarrow \operatorname{im} T = W$.

$\Leftarrow$ Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$ be both injective and surjective:

For all $w \in W$ there exists exactly one $v \in V$ with $T(v) = w$.

Define now a map $S: W \to V$ by $S(w) = v$ (where $T(v) = w$).

Clearly, $ST = I_V$, $TS = I_W$. Need to check, that S is linear!

(and then, $S = T^{-1}$)

·) $S(w_1 + w_2) = S(w_1) + S(w_2)$ for all $w_1, w_2 \in W$:

Since T is linear, $T(S(w_1) + S(w_2)) = T(S(w_1)) + T(S(w_2))$

$= w_1 + w_2$

Then $S(w_1 + w_2) = S(w_1) + S(w_2)$ by definition.

·) $S(aw) = aS(w)$ for $a \in \mathbb{F}$, $w \in W$:

Linearity of T: $T(aS(w)) = aT(S(w)) = aw$.

Then, $S(aw) = aS(w)$ by definition.

$\Rightarrow S$ is linear $\Rightarrow S = T^{-1}$ $\square$

Def 3.58 An invertible linear map is called an isomorphism.

Two vector spaces V, W over $\mathbb{F}$ are called isomorphic, denoted $V \cong W$, if there exists an isomorphism $T \in \mathcal{L}_{\mathbb{F}}(V, W)$.

Prop $T \in \mathcal{L}_{\mathbb{F}}(V, W)$ is an isomorphism if and only if it maps bases in V to bases in W: whenever $\{v_1, \dots, v_n\}$ is a basis for V, then $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.

<u>Proof</u>: $\Rightarrow$ Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$ be invertible, and $\{v_1, \dots, v_n\}$ be a basis for V. To show: $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.

·) linear independence: Let $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ be such that

$$\lambda_1 T(v_1) + \dots + \lambda_n T(v_n) = 0$$

Apply T^{-1} on both sides, then

$$T^{-1}(\lambda_1 T(v_1) + \dots + \lambda_n T(v_n)) = T^{-1}(0)$$

$$\Leftrightarrow \lambda_1 T^{-1}(T(v_1)) + \dots + \lambda_n T^{-1}(T(v_n)) = 0$$

$$\Leftrightarrow \lambda_1 v_1 + \dots + \lambda_n v_n = 0$$

But $\{v_1, \dots, v_n\}$ is a basis and hence linearly independent, so that $\lambda_i = 0$ for $1 \leq i \leq n$ $\Rightarrow \{T(v_1), \dots, T(v_n)\}$ lin. ind.

$\cdot$) $\langle T(v_1), \dots, T(v_n) \rangle = W$.

Since T is surjective, for every $w \in W$ there exists a $v \in V$ with $T(v) = w$. Let $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ be such that

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n .$$

Apply T on both sides of this eq: $w = T(v) = \lambda_1 T(v_1) + \dots + \lambda_n T(v_n)$

$$\Rightarrow w \in \langle T(v_1), \dots, T(v_n) \rangle \ \forall w \in W \Rightarrow W = \langle T(v_1), \dots, T(v_n) \rangle .$$

($\Leftarrow$) Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$ be such that $\{T(v_1), \dots, T(v_n)\}$ is a basis for W whenever $\{v_1, \dots, v_n\}$ is a basis for V.

Define a linear map $S: W \to V$ via $S(T(v_i)) = v_i$ for all $i = 1, \dots, n$.

Since $\{T(v_i)\}_{i=1,\dots,n}$ form a basis, this uniquely defines a linear map S by Prop 3.5 (Lecture 10).

To show: $S = T^{-1}$, or $S(T(v)) = v$ for all $v \in V$.

Let $v \in V$ and $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ s.t. $v = \lambda_1 v_1 + \dots + \lambda_n v_n$.

Set $w = T(v) = \lambda_1 T(v_1) + \dots + \lambda_n T(v_n)$.

$$\begin{aligned} \text{Then } S(w) = S(T(v)) &= S(\lambda_1 T(v_1) + \dots + \lambda_n T(v_n)) \\ &= \lambda_1 S(T(v_1)) + \dots + \lambda_n S(T(v_n)) \\ &= \lambda_1 v_1 + \dots + \lambda_n v_n = v \end{aligned}$$

$$\Rightarrow S = T^{-1} \qquad \square$$

Cor 3.59 Two finite-dim. vector spaces over the same field are isomorphic if and only if they have the same dimension.

<u>Proof:</u> ($\Rightarrow$) Let $T \in \mathcal{L}_{\mathbb{F}}(V,W)$ be an isomorphism, and $\{v_1, \dots, v_n\}$ be a basis for V s.t. $\dim V = n$. Then by the previous Prop., $\{T(v_1), \dots, T(v_n)\}$ is a basis for W, and hence $\dim W = n$.

($\Leftarrow$) Let $n = \dim V = \dim W$, and choose bases $\{v_1, \dots, v_n\}$ for V and $\{w_1, \dots, w_n\}$ for W. Define $T: V \to W$ via $T(v_i) = w_i$ for all $i = 1, \dots, n$ (this defines a unique linear map by Prop 3.5).

$\Rightarrow$ by previous proposition, T is an isomorphism. $\square$

Cor Let V be a vector space over $\mathbb{F}$ with $\dim V = n$.

Then $V \cong \mathbb{F}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_i \in \mathbb{F} \text{ for all } i = 1, \dots, n \right\}$

Cor Let V, W be VS's over $\mathbb{F}$ with $\dim V = n$, $\dim W = m$.

Then $\mathcal{L}_{\mathbb{F}}(V,W) \cong M_{m,n}(\mathbb{F})$, and $\dim \mathcal{L}_{\mathbb{F}}(V,W) = m \cdot n$.

<u>Proof:</u> For fixed bases B_V for V and B_W for W, the isomorphism is given by $\mathcal{M}(\cdot)_{B_V, B_W} : \mathcal{L}_{\mathbb{F}}(V,W) \to M_{m,n}(\mathbb{F})$.

(Simple exercise to show that $\mathcal{M}(\cdot)$ is injective and surjective.) $\square$