Lecture 10: Linear maps

Last time: Dimension of a vector space

$\boxed{\text{Def. 3.2}}$ Linear maps

Let V, W be vector spaces over a field $\mathbb{F}$. A linear map $T: V \to W$ is a function satisfying:

$\cdot)$ $T(x+y) = T(x) + T(y)$ for all $x, y \in V$

$\cdot)$ $T(a \cdot x) = a\,T(x)$ for all $a \in \mathbb{F}$, $x \in V$.

The set of all linear maps from V to W is denoted

$$\mathcal{L}(V,W) = \mathcal{L}_{\mathbb{F}}(V,W).$$

Sometimes we also write $Tv = T(v)$ for $T \in \mathcal{L}(V,W)$, $v \in V$.

Examples of linear maps:

$\cdot)$ Identity map $I: V \to V$, $Iv = v$ for all $v \in V$.

$\cdot)$ Zero map $0: V \to W$, $0v = 0_W$ for all $v \in V$.

$\cdot)$ reflections, rotations, projections, differentiation (see Lecture 1)

·) Fundamental example: Let $m, n \in \mathbb{N}$, and $A = (A_{ij})_{ij}$,

$A_{ij} \in \mathbb{F}$ for $1 \le i \le m$, $1 \le j \le n$.

$$T_A : \mathbb{F}^n \to \mathbb{F}^m, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad y_i = \sum_{j=1}^{n} A_{ij} x_j$$

We will prove later: Every linear map essentially looks like this.

$\boxed{\text{Prop 3.11}}$ Every linear map $T : V \to W$ satisfies $T(0_V) = 0_W$.

<u>Proof</u>: We have $0 = 0 + 0$, and hence

$$T(0) = T(0+0) = T(0) + T(0) \Rightarrow T(0_V) = 0_W. \qquad \square$$

A linear map is uniquely defined on a basis:

$\boxed{\text{Prop 3.5}}$ Let $\{v_1, \dots, v_n\}$ be a basis of V, and let $w_1, \dots, w_n \in W$ be arbitrary. Then there exists a unique linear map $T : V \to W$ with $T(v_j) = w_j$ for all $1 \le j \le n$.

<u>Proof</u>: ·) Existence: Define a map $T : V \to W$ via $T(v_j) = w_j \ \forall j$, and extend it linearly to the whole space:

for $v \in V$, $v = \sum_{j=1}^{n} a_i v_i$ ($\{v_i\}_i$ are a basis of V),

$$T(v) = T\left(\sum_{j=1}^{n} a_i v_i\right) := \sum_{j=1}^{n} a_j T(v_j) = \sum_{j=1}^{n} a_j w_j$$

This is a linear map by definition, and it is well-defined, because the coefficients $a_j \in \mathbb{F}$ in $v = \sum_{j=1}^{n} a_j v_j$ are unique, (Prop 2.29) and if $v = w$, then $v - w = 0$, and

$$\underset{\substack{\uparrow \\ \text{Prop 3.11}}}{0 =} T(0) = T(v-w) = T(v) - T(w) \Rightarrow T(v) = T(w).$$

- Uniqueness: Let S be another linear map $V \to W$ with $S(v_j) = w_j \ \forall j$

  Let $v = \sum_{j=1}^{n} a_j v_j \in V$ be arbitrary, then

$$T(v) = T\left(\sum_{j=1}^{n} a_j v_j\right) = \sum_{j=1}^{n} a_j T(v_j) = \sum_{j=1}^{n} a_j w_j = \sum_{j=1}^{n} a_j S(v_j)$$

$$= S\left(\sum_{j=1}^{n} a_j v_j\right) = S(v)$$

$\Rightarrow$ since v was arbitrary, $T = S$. $\square$

The set of all linear maps $\mathcal{L}(V,W)$ is itself a vector space:

$\boxed{\text{Prop 3.6/3.7}}$

Define addition and scalar multiplication in $\mathcal{L}(V,W)$ point-wise:

- $S, T \in \mathcal{L}(V,W)$: $S+T: v \mapsto S(v) + T(v)$
- $a \in \mathbb{F}$, $T \in \mathcal{L}(V,W)$: $aT: v \mapsto aT(v)$

(HW: $S+T,\ aT \in \mathcal{L}(V,W)$) Then $(\mathcal{L}_{\mathbb{F}}(V,W), +, \cdot)$ is a VS over $\mathbb{F}$.

Proof: Easy exercise (neutral element for $+$: zero map $0: v \mapsto 0_V$).

$\square$

We can also compose / "multiply" linear maps:

$\boxed{\text{Def 3.8}}$ Let $T \in \mathcal{L}_{\mathbb{F}}(U, V)$ and $S \in \mathcal{L}_{\mathbb{F}}(V, W)$,

then the map $S \circ T = ST : U \to W$, defined by

$$(ST)(u) = S(T(u)) \quad \text{for } u \in U,$$

is again a linear map:

$$\cdot)\ (ST)(x+y) = S(T(x+y)) = S(T(x)+T(y))$$
$$= S(T(x)) + S(T(y))$$
$$= (ST)(x) + (ST)(y) \quad \forall x, y \in U$$

$$\cdot)\ (ST)(ax) = S(T(ax)) = S(aT(x))$$
$$= a\, S(T(x)) = a(ST)(x) \quad \forall a \in \mathbb{F},\ x \in U$$

Prop 3.9 Properties of the product of linear maps

i) Associativity: $(RS)T = R(ST)$ for $T \in \mathcal{L}(V_1, V_2)$

$S \in \mathcal{L}(V_2, V_3)$

$R \in \mathcal{L}(V_3, V_4)$

ii) neutral element: $T I_V = I_W T$ for all $T \in \mathcal{L}(V, W)$

$(I_X : X \to X$ identity map$)$

iii) Distributive laws: $(Q+R)S = QS + RS$ $\quad Q, R \in \mathcal{L}(V, W)$

$Q(S+T) = QS + QT$ $\quad S, T \in \mathcal{L}(U, V)$

Proof: i) $x \in V_1$: $[(RS)T](x) = (RS)(T(x))$

$$= R(S(T(x)))$$

$$= R(ST(x))$$

$$= [R(ST)](x) \Rightarrow (RS)T = R(ST).$$

ii) $x \in V$: $(TI_V)(x) = T(I_V(x)) = T(x) = I_W(T(x)) = (I_W T)(x)$

$$\Rightarrow TI_V = I_W T.$$

iii) HW

$\square$