Lecture 8: Bases

Last time: Span, linear independence

Cor 2.26 Every subspace of a finite-dimensional vector space is itself finite-dimensional.

Proof: Let $U \leq V$, V finite-dim. (i.e., it has a finite spanning set).
Assume $U \neq \{0\}$ (because otherwise we're done).

Now choose vectors u_j inductively such that at each step the list $\{u_1, \dots, u_j\}$ is linearly independent.
Do this until $U = \langle u_1, \dots, u_m \rangle$ for some m.
The list $\{u_1, \dots, u_m\}$ is linearly indep. in U, and hence also lin. indep. in V.
$\Rightarrow$ By Prop 2.23 m is at most equal to the length of a spanning list for V $\Rightarrow$ m is finite $\Rightarrow$ U fin.-dim.
$\square$

Def 2.27 Basis of a vector space

A basis of a vector space V is a set of vectors that is

- linearly independent
- spans V.

A basis consists of ...

·) the <u>minimal</u> number of vectors spanning a vector space.

·) the <u>maximal</u> number of linearly independent vectors.

<u>Ex.:</u> ·) Standard basis $\{e_1, \ldots, e_n\}$ is a <u>basis</u> for $\mathbb{F}^n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots \; e_i := \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Clearly, $\{e_1, \ldots, e_n\}$ are linearly independent, and they span $\mathbb{F}^n$:
$$\mathbb{F}^n \ni x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 + \ldots + x_n e_n.$$

·) $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$: $v_3 = v_1 + v_2$

$\{v_1, v_2, v_3\}$ is <u>not</u> a basis, but $\{v_i, v_j\}_{i \neq j}$ <u>is</u> a basis for $\mathbb{R}^2$ (HW3)

·) $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ do <u>not</u> span $\mathbb{R}^3$ $\left(v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \notin \langle v_1, v_2 \rangle \right)$

$\{v_1, v_2\}$ are lin. indep., but not a basis for $\mathbb{R}^3$.

($\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3 \rightarrow$ HW3)

Prop 2.29 A set $\{v_1, \ldots, v_m\}$ of vectors $v_i \in V$ is a basis if and only if every $w \in V$ can be written as

$$w = a_1 v_1 + \ldots + a_m v_m \quad \text{for \underline{unique} scalars } a_i \in \mathbb{F}.$$

Proof: ($\Rightarrow$) Let $w = a_1 v_1 + \ldots + a_m v_m = b_1 v_1 + \ldots + b_m v_m \in V$

$a_i, b_i \in \mathbb{F}$. Then $(a_1 - b_1) v_1 + \ldots + (a_m - b_m) v_m = 0$

$\Rightarrow$ by linear independence of $\{v_1, \ldots, v_m\}$, $a_i - b_i = 0$ for all i,

or $a_i = b_i$ for all i.

($\Leftarrow$) Suppose for all $v \in V$ there are unique elements $a_i \in \mathbb{F}$

s.t. $v = a_1 v_1 + \ldots + a_m v_m \Rightarrow V = \langle v_1, \ldots, v_m \rangle$.

Then $0 = a_1 v_1 + \ldots + a_m v_m = 0 v_1 + \ldots + 0 v_m$

$\Rightarrow$ by uniqueness, $a_i = 0$ for all i

$\Rightarrow \{v_1, \ldots, v_m\}$ are linearly independent.

$\Rightarrow \{v_1, \ldots, v_m\}$ are a basis for V. $\square$

Prop 2.31 Let $V = \langle v_1, \ldots, v_m \rangle$ for $v_i \in V$.

Then there is a subset of $\{v_1, \ldots, v_m\}$ of size $n \leq m$ that is a basis for V.

Proof: Order $\{v_1, \dots, v_m\}$ such that $v_1 \neq 0$.

(if all $v_i = 0$, then $V = \{0\}$, and there is nothing to show)

Now for each $j = 2, \dots, m$ we look at v_j and either

·) $v_j \in \langle v_1, \dots, v_{j-1} \rangle$ and we drop v_j, or

·) add it to the list.

The resulting list $\{v_1, v_{i_2}, \dots, v_{i_n}\}$ is lin. independent by construction.
and this list still spans all of V, since we only remove vectors that are already in the span $\Rightarrow$ $\{v_1, v_{i_2}, \dots, v_{i_n}\}$ is a basis of V.

by Prop 2.23, $n \leq m$. $\square$

$\boxed{\text{Corollary 2.23}}$ Every finite-dimensional vector space has a basis.

Dually, we can always extend a set of lin. indep. vectors to a basis:

Prop 2.33 Every set $\{v_1, \dots, v_n\}$ of linearly indep. vectors in V can be extended to a basis of V.

Proof: Let $\{w_1, \dots, w_m\}$ be some vectors spanning V.
Consider the (ordered) list of vectors $v_1, \dots, v_n, w_1, \dots, w_m$.
The same procedure as in the proof of 2.31 will remove some of the w_j's without changing the span of the list.
We will never remove any of the v_i's because they are lin. independent. $\Rightarrow$ arrive at a basis $\{v_1, \dots, v_n, w_{i_1}, \dots, w_{i_k}\}$ for V. $\square$

Application: subspaces as direct summands

Prop 2.34 Let V be a finite-dim. vector space and $U \leq V$ be a subspace. Then there is another subspace $W \leq V$ s.t. $V = U \oplus W$.

Proof: Since V is finite-dim., so is U by Cor. 2.26.
Let $\{u_1, \dots, u_m\}$ be a basis for U. Since $\{u_1, \dots, u_m\}$ is linearly indep. in U and also in V, we can extend it to a basis $\{u_1, \dots, u_m, w_1, \dots, w_n\}$ of V by Prop. 2.33.

Define $W = \langle w_1, \dots, w_n \rangle \leq V$.

Claim: $V = U \oplus W$

To show (Lecture 6): a) $V = U + W$

b) $U \cap W = \{0\}$

a) recall that $\{u_1, \dots, u_m, w_1, \dots, w_n\}$ are a basis for V

$\Rightarrow \forall v \in V: \quad v = a_1 u_1 + \dots + a_m u_m + b_1 w_1 + \dots + b_n w_n$

for some $a_i, b_j \in \mathbb{F}$.

since $\sum_{i=1}^{m} a_i u_i \in U$, and $\sum_{j=1}^{n} b_j w_j \in W$

$\Rightarrow V = U + W$.

b) $v \in U \cap W$, $\quad v = a_1 u_1 + \dots + a_m u_m = b_1 w_1 + \dots + b_n w_n$

$$a_1 u_1 + \dots + a_m u_m - b_1 w_1 - \dots - b_n w_n = 0$$

by linear independence of $\{u_1, \dots, u_m, w_1, \dots, w_n\}$,

$a_i = 0$ for $i = 1, \dots, m$ and $b_j = 0$ for $j = 1, \dots, n$

$\Rightarrow v = 0 \quad \Rightarrow \quad U \cap W = \{0\}$.

a) + b) $\Rightarrow V = U \oplus W$. $\square$