Lecture 7: Span and linear independence

Last time: Subspaces, sums of subspaces, direct sums

$\boxed{\text{Def (2.3, 2.5)}}$ Let V be a vector space over a field $\mathbb{F}$, and $v_1, \dots, v_m \in V$.

·) A linear combination of $v_1, \dots, v_m$ is a vector

$$\sum_{i=1}^{m} a_i v_i, \quad \text{where } a_i \in \mathbb{F} \text{ for } 1 \le i \le m.$$

·) The span $\langle v_1, \dots, v_m \rangle$ of the vectors v_i is the set of all linear combinations: $\langle v_1, \dots, v_m \rangle = \left\{ \sum_{i=1}^{m} a_i v_i : a_i \in \mathbb{F} \text{ for } 1 \le i \le m \right\}$

By definition, $\langle \emptyset \rangle = \{0\}$.

$\boxed{\text{Prop 2.7}}$ Let $v_1, \dots, v_m \in V$. Then $\langle v_1, \dots, v_m \rangle$ is a subspace of V, and it is the smallest subspace of V containing $v_1, \dots, v_m$.

<u>Proof</u>: We first show that $\langle v_1, \dots, v_m \rangle \le V$.

We have $0 = 0v_1 + \dots + 0v_m \in \langle v_1, \dots, v_m \rangle$

Clearly, the set of all linear combinations of $v_1, \dots, v_m$ is closed under vector addition and scalar multiplication by definition.

$\Rightarrow$ by Prop 1.34, $\langle v_1, \dots, v_m \rangle \le V$ is a subspace.

Let now $W \subseteq V$ with $v_i \in W$ for $1 \leq i \leq m$.

By Prop 1.34, W is closed under add. and scalar mult.,

and hence $\sum_{i=1}^{m} a_i v_i \in W$ for all $a_i \in \mathbb{F}$, $1 \leq i \leq m$.

Therefore, $v_j \in \langle v_1, \dots, v_m \rangle \subseteq W \Rightarrow$ claim. $\square$

Def 2.8 A vector space V is called finite-dimensional, if it has a finite spanning set: $\exists v_1, \dots, v_m \in V$, $m \in \mathbb{N}$, with $\langle v_1, \dots, v_m \rangle = V$.

<u>Ex.:</u>
- For any field $\mathbb{F}$ and $n \in \mathbb{N}$, the vector space $\mathbb{F}^n$ is finite-dimensional, because it is spanned by the standard basis $e_1, \dots, e_n$, where $(e_i)_j = \delta_{ij}$ for $1 \leq i, j \leq n$. $\mathbb{F}^n = \langle e_1, \dots, e_n \rangle$.
- $P_d(\mathbb{F})$ (polynomials over $\mathbb{F}$ with $\deg \leq d$) is finite-dim, since $P_d(\mathbb{F}) = \langle 1, x, x^2, \dots, x^d \rangle$ $(1 \in \mathbb{F})$.
- The space of all polynomials over $\mathbb{F}$ (with unbounded degree) is <u>not</u> finite-dimensional.
- The space $C([a,b])$ of cont. functions on $[a,b]$ is <u>not</u> fin.-dim.

Linear (in-) dependence

Consider $\mathbb{R}^2$ and the vectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

This list of vectors is in some sense redundant, because

- e.g., $v_3 = v_1 + v_2$ or $v_1 + v_2 - v_3 = 0$
- $\langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle$ but $\langle v_i \rangle \subsetneq \langle v_1, v_2 \rangle$, $i = 1, 2$

We call $\{v_1, v_2, v_3\}$ linearly dependent, and $\{v_1, v_2\}$ lin. independent.

Def 2.17 Linear independence

A list of $\{v_1, \dots, v_m\}$ of vectors $v_i \in V$ is called linearly independent, if

$$a_1 v_1 + \dots + a_m v_m = 0 \text{ for } a_i \in \mathbb{F} \Rightarrow a_i = 0 \text{ for } 1 \leq i \leq m.$$

By definition, the empty list $\{\}$ is linearly independent.

If $\{v_1, \dots, v_m\}$ is <u>not</u> linearly independent, we call it linearly dependent, i.e., there are $a_i \in \mathbb{F}$, $a_i \neq 0$ for at least one $i \in \{1, \dots, m\}$, s.t. $\sum_{j=1}^{m} a_j v_j = 0 \Leftrightarrow v_i = \sum_{\substack{j=1 \\ j \neq i}}^{m} -\frac{a_j}{a_i} v_j$

$\boxed{\text{Lem } 2.21}$ Let $\{v_1, \dots, v_m\}$ with $v_i \in V$ be lin. dependent.

Then there exists $j \in \{1, \dots, m\}$ such that

i) $v_j \in \langle v_1, \dots, v_{j-1} \rangle$

ii) $\langle v_1, \dots, v_m \rangle = \langle v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m \rangle$

<u>Proof</u>: By assumption, there are $a_i \in \mathbb{F}$, not all $= 0$, s.t.

$$a_1 v_1 + \dots + a_m v_m = 0. \qquad (*)$$

$$\Downarrow$$

$$a_1 v_1 + \dots + a_j v_j = 0$$

Let $j \in \{1, \dots, m\}$ be the longest index s.t. $a_j \neq 0$.

Then we can rewrite $(*)$ as $v_j = -\frac{a_1}{a_j} v_1 - \dots - \frac{a_{j-1}}{a_j} v_{j-1} \qquad (**)$

$$\Rightarrow v_j \in \langle v_1, \dots, v_{j-1} \rangle \Rightarrow \text{i)}$$

ii) Let $w \in \langle v_1, \dots, v_m \rangle$, i.e., $\exists c_i \in \mathbb{F}$ s.t.

$$w = \sum_{i=1}^{m} c_i v_i = c_j v_j + \sum_{i \neq j} c_i v_i$$

$$= \sum_{i=1}^{j-1} \left(c_i - \frac{c_i a_i}{a_j} \right) v_i + \sum_{i=j+1}^{m} c_i v_i$$

$$\Rightarrow w \in \langle v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m \rangle \Rightarrow \text{ii)}. \qquad \square$$

$\boxed{\text{Prop 2.23}}$ Let V be a finite-dim. vector space, i.e., there are $v_1, \dots, v_n \in V$ s.t. $\langle v_1, \dots, v_n \rangle = V$.
Let also $u_1, \dots, u_m \in V$ s.t. $\{u_1, \dots, u_m\}$ are lin. independent.
Then $m \leq n$.

Proof: Since $V = \langle v_1, \dots, v_n \rangle$, we have $u_1 \in \langle v_1, \dots, v_n \rangle$.
$\Rightarrow \{u_1, v_1, \dots, v_n\}$ is linearly dependent.
By Lem. 2.21, there is a $j \in \{1, \dots, n\}$ s.t. we can remove v_j from the list $u_1, v_1, \dots, v_m$ and still have

$$\langle u_1, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n \rangle = V.$$

Assume we have done this $k-1$ times:

$$\langle u_1, \dots, u_{k-1}, v_{i_1}, \dots, v_{i_{n-k+1}} \rangle = V$$

Now add u_k to this list, reorder it to $u_1, \dots, u_k, v_{i_1}, \dots, v_{i_{n-k+1}}$.
By Lem 2.21, we can again remove one of the v_j's.
Since $\{u_1, \dots, u_k\}$ are part of a lin. independent set of vectors, we always remove one of the v_j's in this process.
Continue this process until $k = m$, so that we have inserted all u_i's in the list. In each step, we found one v_j vector to remove $\Rightarrow \# u_i\text{'s} \leq \# v_j\text{'s} \Rightarrow m \leq n$. $\square$