Lecture 3: Geometry of linear systems

Last time: Gaussian elimination, (reduced) row-echelon form

Ex.: $3x_1 + 4x_2 = 6$

$3x_1 + 4x_2 = 7$

$$\longrightarrow \left(\begin{array}{cc|c} 3 & 4 & 6 \\ 3 & 4 & 7 \end{array}\right) \xrightarrow[\text{elim.}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 4/3 & 2 \\ 0 & 0 & 1 \end{array}\right)$$

$$\Longleftrightarrow \quad \begin{aligned} x_1 + \tfrac{4}{3} x_2 &= 2 \\ 0 &= 1 \ \lightning. \end{aligned}$$

system is inconsistent (no solutions)

$\Longleftrightarrow$ RREF has a pivot in the last column.

Sometimes we have infinitely many solutions:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 2 \\ x_3 - 3x_4 &= 3 \\ x_5 &= 4 \end{aligned} \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right) \ (\text{RREF!})$$

pivots

Turn those variables, whose columns do not have pivot elements,

into parameters: $x_2 \equiv t_1$, $x_4 \equiv t_2$, $t_1, t_2 \in \mathbb{R}$

$$\left.\begin{aligned} x_1 &= 2 - 2t_1 - t_2 \\ x_3 &= 3 + 3t_2 \\ x_5 &= 4 \end{aligned}\right\} \quad \text{solutions: } \left\{ \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 4 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} : t_1, t_2 \in \mathbb{R} \right\}$$

2-dim. affine solution space in $\mathbb{R}^5$

Summary: system of linear eq's $\leftrightarrow$ augmented matrix of coeff's.

$\leftrightarrow$ (reduced) row-echelon form

$$\begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{matrix} \longleftrightarrow \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right) \longleftrightarrow \left(\begin{array}{cccc|c} 1 & & 0 & \vdots & c_1 \\ 0 & \dots & 1 & \vdots & \vdots \\ \vdots & & 0 & \vdots & \vdots \\ 0 & & \vdots & & c_m \end{array}\right)$$

Observations: System of linear eq's ...

- ·) is inconsistent (no solutions) $\Leftrightarrow$ RREF has a pivot in last column.

- ·) has a unique solution $\Leftrightarrow$ RREF has a pivot in every column except the last one

- ·) has multiple solutions $\Leftrightarrow$ the last column and at least one other column in the RREF do not have pivots.

Goal today: develop a geometric intuition of linear eq's and their solutions.

First lecture: Euclidean space $\mathbb{R}^n$: vectors $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $v_i \in \mathbb{R}$, that we can add and multiply by real scalars.

$\boxed{\text{Def}}$ Let $v_1, \dots, v_m \in \mathbb{R}^n$.

·) A linear combination of $v_1, \dots, v_m$ is a vector

$$\sum_{i=1}^{m} c_i v_i \quad \text{where } c_i \in \mathbb{R} \ \forall i = 1, \dots, m.$$

·) The span $\langle v_1, \dots, v_m \rangle$ of $v_1, \dots, v_m$ is the set of all linear combinations of $v_1, \dots, v_m$:

$$\langle v_1, \dots, v_m \rangle = \left\{ \sum_{i=1}^{m} c_i v_i : c_i \in \mathbb{R} \text{ for all } i = 1, \dots, m \right\}.$$

We can ask: Given $v_1, \dots, v_m \in \mathbb{R}^n$, is $w \in \langle v_1, \dots, v_m \rangle$?

$$\Leftrightarrow \exists c_i \in \mathbb{R} \text{ s.t. } w = \sum_{i=1}^{m} c_i v_i \ ?$$

Ex: $v_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $w = \begin{pmatrix} 39 \\ 34 \\ 26 \end{pmatrix}$

Is $w \in \langle v_1, v_2, v_3 \rangle$? $\Leftrightarrow \exists c_1, c_2, c_3$ s.t. $w = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\Longleftrightarrow \begin{pmatrix} 39 \\ 34 \\ 26 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\Longleftrightarrow \underbrace{\begin{aligned} 3c_1 + 2c_2 + c_3 &= 39 \\ 2c_1 + 3c_2 + c_3 &= 34 \\ c_1 + 2c_2 + 3c_3 &= 26 \end{aligned}}_{\text{System of linear eq's from Lect. 2!}}$$

Ex.: any vector $v \in \mathbb{R}^n$ can be written as

$$v = v_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{=: e_1} + v_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{=: e_2} + \dots + v_n \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{=: e_n} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$\{e_1, \dots, e_n\}$ is called the "standard basis" (we will define "basis" later!)

Clearly, $\langle e_1, \dots, e_n \rangle = \mathbb{R}^n$

Important: Different linear combinations can lead to the same vec:

Ex: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Then: $w = v_1 + 2 v_2 = v_2 + v_3$

$\Rightarrow w \in \langle v_1, v_2, v_3 \rangle$ but there is no unique way of writing w in terms of v_1, v_2, v_3.

Moral of the story: Solving linear equations

$\Leftrightarrow$ writing a given vector as a linear combination of some vectors.

Ex.: $$\left.\begin{aligned} x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_4 &= 20 \\ 2x_1 + \tfrac{1}{4}x_2 \qquad\quad &= 36 \\ x_1 + 8x_3 + x_4 &= 176 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= 16 + \tfrac{1}{2}t \\ x_2 &= 16 - 4t \\ x_3 &= 20 - \tfrac{3}{16}t \\ x_4 &= t \quad , \quad t \in \mathbb{R} \end{aligned}$$

solutions: $$\ell : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 16 \\ 20 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1/2 \\ -4 \\ -3/16 \\ 1 \end{pmatrix}$$

line in $\mathbb{R}^4$ through $\begin{pmatrix} 16 \\ 16 \\ 20 \\ 0 \end{pmatrix}$ along $\begin{pmatrix} 1/2 \\ -4 \\ -3/16 \\ 1 \end{pmatrix}$

Geometrically, the (3-dim) hyperplanes

$$\left.\begin{aligned} \varepsilon_1 &: x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{2}x_4 = 20 \\ \varepsilon_2 &: 2x_1 + \tfrac{1}{4}x_2 \qquad\quad = 36 \\ \varepsilon_3 &: x_1 + 8x_3 + x_4 = 176 \end{aligned}\right\}$$ intersect in the line ℓ above.

Ex.:

$$\left.\begin{aligned} x_1 + 2x_2 + 5x_4 &= 3 \\ -x_1 - 2x_2 + x_3 - 6x_4 + x_5 &= 2 \\ -2x_1 - 4x_2 - 10x_4 + x_5 &= 8 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -3 - 2t_1 - 5t_2 \\ x_2 &= t_1 \\ x_3 &= 1 + t_2 \\ x_4 &= t_2 \\ x_5 &= -2 \end{aligned} \qquad t_1, t_2 \in \mathbb{R}$$

Solutions: ε: $\begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} + t_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

2-dim. plane in $\mathbb{R}^5$ = intersection of 3 4-dim. hyperplanes in $\mathbb{R}^5$.