MATH 416 Abstract Linear Algebra

Homework Week 5 – September 23, 2021

Exercise 1 (6 points): Injectivity and surjectivity
Let V be a (finite-dim.) vector space over a field $\mathbb{F}$. Let $v_1, \ldots, v_m \in V$ and define the linear map

$$T\colon \mathbb{F}^m \to V, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \mapsto \sum_{i=1}^{m} x_i v_i.$$

(i) Prove that T is injective if and only if $\{v_1, \ldots, v_m\}$ are linearly independent.

(ii) Prove that T is surjective if and only if $V = \langle v_1, \ldots, v_m \rangle$.

Exercise 2 (4 points): Linear maps as matrices I
Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$, and fix bases $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ for V and $\mathcal{B}_W = \{w_1, \ldots, w_m\}$ for W. In the following, we abbreviate $\mathcal{M}(\cdot) = \mathcal{M}(\cdot)_{\mathcal{B}_V, \mathcal{B}_W}$.
Show that:

(i) $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$ for $S, T \in \mathcal{L}_{\mathbb{F}}(V, W)$.

(ii) $\mathcal{M}(aT) = a\mathcal{M}(T)$ for $a \in \mathbb{F}$ and $T \in \mathcal{L}_{\mathbb{F}}(V, W)$.

Exercise 3 (4 points): Linear maps as matrices II
Consider the following linear map:

$$T\colon \mathbb{R}^3 \to \mathbb{R}^4, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 + x_3 \\ x_1 - x_2 - x_3 \\ x_1 + x_2 \\ x_2 - x_3 \end{pmatrix}$$

(i) Determine $\mathcal{M}(T)_{\mathcal{S}_3, \mathcal{S}_4}$, where $\mathcal{S}_3$ and $\mathcal{S}_4$ are the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

(ii) Let now

$$\mathcal{B}_V = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \mathcal{B}_W = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

and determine $\mathcal{M}(T)_{\mathcal{B}_V, \mathcal{B}_W}$.

Remark: You do not need to show that $\mathcal{B}_V$ and $\mathcal{B}_W$ are indeed bases for $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.