MATH 416 Abstract Linear Algebra

Homework Week 3 – September 9, 2021

Exercise 1 (3 points): Linear independence and span

(i) (2 points) Let $z_1 = 1 + i$ and $z_2 = 1 - i$. First consider the complex numbers $\mathbb{C}$ as a vector space over the field $\mathbb{R}$, and show that $\{z_1, z_2\}$ is linearly independent over $\mathbb{R}$. Then consider $\mathbb{C}$ as a vector space over itself (i.e., $\mathbb{F} = \mathbb{C}$), and show that now $\{z_1, z_2\}$ is linearly dependent.

(ii) (1 point) Let $\{v_1, \ldots, v_m\}$ be a set of linearly independent vectors in V, and let $w \in V$. Show that, if $\{v_1 + w, \ldots, v_m + w\}$ are linearly dependent, then $w \in \langle v_1, \ldots v_m \rangle$.

Exercise 2 (3 points): Polynomials

(i) (2 points) Show that the set $\{1 + x + x^2, x + x^2, x^2\}$ is a basis for $P_2(\mathbb{F})$.

(ii) (1 point) Let $P(\mathbb{F}) = \bigcup_{d \in \mathbb{N}} P_d(\mathbb{F})$ be the vector space of all polynomials of arbitrary (i.e., unbounded) degree with coefficients in $\mathbb{F}$. Prove that $P(\mathbb{F})$ is not finite-dimensional.

Hint: Use proof by contradiction by assuming that $P(\mathbb{F})$ is finite-dimensional, i.e., assume that there are finitely many polynomials $p_1, \ldots, p_m$ such that $P(\mathbb{F}) = \langle p_1, \ldots, p_m \rangle$. Now find a polynomial in $P(\mathbb{F})$ that lies outside the span of $\{p_1, \ldots, p_m\}$.

Exercise 3 (3 points): Bases I

(i) Let $\{u_1, u_2, u_3\}$ be the following vectors in $\mathbb{R}^2$:

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Show that $\{u_1, u_2, u_3\}$ is not a basis of $\mathbb{R}^2$, but $\{u_i, u_j\}$ is a basis for any $1 \leq i < j \leq 3$.

(ii) Prove that the following set of vectors $\{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(iii) Prove that the following set of vectors $\{w_1, w_2, w_3\}$ does not form a basis of $\mathbb{R}^3$:

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

Exercise 4 (3 points): Bases II

Let U be the subspace of $\mathbb{R}^5$ defined by[1]

$$U = \left\{ (x_1, x_2, x_3, x_4, x_5)^T \in \mathbb{R}^5 \colon x_1 = 3x_2 \text{ and } x_3 = 7x_4 \right\}$$

(i) Find a basis for U.

(ii) Extend the basis you found in (i) to a basis of $\mathbb{R}^5$.

(iii) Find a subspace $W \leq \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

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[1] Here, ^{T} denotes transposition, and $x = (x_1, x_2, x_3, x_4, x_5)^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.