MATH 416 Abstract Linear Algebra

Homework Week 2 – September 2, 2021

Last update: September 5, 2021

Exercise 1 (3 points): Ordered fields

An *ordered field* is a field $\mathbb{F}$ together with a total order[1] $\prec$ satisfying the following properties for all $a, b, c \in \mathbb{F}$: (i) If $a \prec b$, then $a + c \prec b + c$. (ii) If $0 \prec a$ and $0 \prec b$, then $0 \prec ab$.

(i) Show that $0 \prec 1$ in every ordered field, where $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ are the additive and multiplicative neutral elements in the field, respectively.
Hint: Assume that $1 \prec 0$ and show that this leads to a contradiction.

(ii) Show that squares are non-negative in every ordered field, i.e., $0 \preceq a^2$ for all $a \in \mathbb{F}$.
Hint: Look at the three possible cases $0 \prec a$, $0 = a$, and $a \prec 0$.

(iii) Use (i) and (ii) to show that the complex numbers cannot be turned into an ordered field.

Exercise 2 (3 points): Complex conjugate and roots of real polynomials

(i) Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ for any $z_1, z_2 \in \mathbb{C}$.

(ii) Show that, for any $z \in \mathbb{C}$, we have $z = \overline{z}$ if and only if $z \in \mathbb{R}$.

(iii) Use (i) and (ii) to show the following statement: Let $p = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial over the reals, i.e., $a_i \in \mathbb{R}$ for $0 \leq i \leq n$. Then $z \in \mathbb{C}$ is a root of p (i.e., $p(z) = 0$) if and only if $\overline{z}$ is a root of p.

Exercise 3 (6 points): Vector spaces

Let S be an arbitrary set, $\mathbb{F}$ an arbitrary field, and consider the space $\mathbb{F}^S = \{f : S \to \mathbb{F}\}$ of functions from S to $\mathbb{F}$. Recall that addition and scalar multiplication in $\mathbb{F}^S$ are defined point-wise: for $f, g \in \mathbb{F}^S$ and $a \in \mathbb{F}$ the functions $f + g$ and af are defined via $(f + g)(s) = f(s) + g(s)$ and $(af)(s) = af(s)$ for $s \in S$, respectively.

(i) (3 points) Show that $\mathbb{F}^S$ is a vector space by checking that addition and scalar multiplication as defined above satisfy the axioms (V1)-(V4), (S1), (S2), (D1) and (D2) of Definition 1.19 in the lecture.

[1] A *partial order* $\preceq$ on a set S is a binary operation that for all elements $r, s, t \in S$ satisfies i) reflexivity: $r \preceq r$; ii) antisymmetry: if $r \preceq s$ and $s \preceq r$, then $r = s$; iii) transitivity: if $r \preceq s$ and $s \preceq t$, then $r \preceq t$. Two elements $s_1, s_2 \in S$ may be *incomparable*, i.e., neither $s_1 \preceq s_2$ nor $s_2 \preceq s_1$ holds. A *total order* is a partial order for which every two elements are comparable, i.e., for any $s_1, s_2 \in S$ we have $s_1 \preceq s_2$ or $s_2 \preceq s_1$. Given a (non-strict) order $\preceq$ on a set S, we can always define a *strict* order $\prec$ via $r \prec s :\Leftrightarrow r \preceq s$ and $r \neq s$ for $r, s \in S$.

(ii) (2 points) Let now S be a finite set with cardinality $|S| = k \in \mathbb{N}$. Show that every function $f \in \mathbb{F}^S$ can be written as a linear combination of "elementary functions" f_t for $t \in S$. These functions are defined via

$$f_t \colon S \longrightarrow \mathbb{F}, \quad s \longmapsto \begin{cases} 1 & \text{if } s = t \\ 0 & \text{else.} \end{cases}$$

More compactly, $f_t(s) = \delta_{st}$.

(iii) (1 point) Use (ii) to show that, for finite S with $|S| = k \in \mathbb{N}$, the vector space of functions $\mathbb{F}^S$ can be identified with the vector space of column vectors $\mathbb{F}^k$.
Hint: There's a 1-1 correspondence between the elementary functions f_t from (ii) and the "standard basis" of $\mathbb{F}^k$ that we defined in the lecture:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad e_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Using this correspondence, set up a 1-1 correspondence between functions in $\mathbb{F}^S$ and vectors in $\mathbb{F}^k$.

Exercise 4 (3 points): Subspaces

(i) Let $U, W \leq V$ be subspaces of a vector space V. Show that $U \cap W$ is also a subspace of V.

(ii) Prove that the subset $\{x \in \mathbb{F}^3 \colon x_1 x_2 x_3 = 0\} \subset \mathbb{F}^3$ is *not* a subspace of $\mathbb{F}^3$.

(iii) Consider the subspace

$$U = \left\{ \begin{pmatrix} x \\ x \\ y \end{pmatrix} : x, y \in \mathbb{F} \right\} \leq \mathbb{F}^3.$$

Find another subspace $W \leq \mathbb{F}^3$ such that $\mathbb{F}^3 = U \oplus W$.