MATH 416 Abstract Linear Algebra

Homework Week 13 – November 20, 2021

Exercise 1 (6 points):
Find a basis for $\mathbb{C}^3$ consisting of generalized eigenvectors for:

$$\text{(i)} \quad S\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x-2y \\ 3y-2z \\ z \end{pmatrix} \qquad\qquad \text{(ii)} \quad T\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

Exercise 2 (6 points):
Find the characteristic polynomial p_T and the minimal polynomial q_T for each matrix below.

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 3 (3 points):

(i) Let N be a nilpotent operator. Prove that 0 is the only eigenvalue of N.

(ii) Let $T \in \mathcal{L}(V)$. Show that

$$V = \operatorname{im} T^0 \supseteq \operatorname{im} T \supseteq \operatorname{im} T^2 \supseteq \cdots \supseteq \operatorname{im} T^k \supseteq \operatorname{im} T^{k+1} \supseteq \ldots$$