MATH 416 Abstract Linear Algebra

Homework Week 10 – October 28, 2021

Exercise 1 (2 points): Orthogonal complements and Gram-Schmidt procedure
Let $\{u_1, \ldots, u_m\}$ be a basis for a subspace U of an inner product space V, and let

$$\mathcal{B} = \{u_1, \ldots, u_m, w_1, \ldots, w_n\}$$

be a basis for V. Prove that applying the Gram-Schmidt procedure to $\mathcal{B}$ yields an orthonormal basis (ONB) $\{\hat{u}_1, \ldots, \hat{u}_m, \hat{w}_1, \ldots, \hat{w}_n\}$, where $\{\hat{u}_1, \ldots, \hat{u}_m\}$ is an ONB for U and $\{\hat{w}_1, \ldots, \hat{w}_n\}$ is an ONB for the orthogonal complement $U^\perp$ of U.

Exercise 2 (7 points): Inner products and orthogonal complements

(i) (4 points) Let $U \leq \mathbb{R}^4$ be the subspace spanned by the vectors $v_1 = (1, 2, 3, -4)^T$ and $v_2 = (-5, 4, 3, 2)^T$. Find orthonormal bases for U and its orthogonal complement $U^\perp$ for the standard inner product $\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$.

(ii) (3 points) Consider the following inner product on $\mathbb{R}^3$: $\langle x, y \rangle_{\text{alt}} := 2x_1y_1 + x_2y_2 + 2x_3y_3$. Compute $\{v\}^\perp$ for the vector $v = (1, -2, 1)^T \in \mathbb{R}^3$.

Warning: If you use the Gram-Schmidt (GS) procedure for this example, then you need to use the inner product $\langle x, y \rangle_{\text{alt}}$ and the associated norm $\|x\|_{\text{alt}} := \sqrt{\langle x, x \rangle_{\text{alt}}}$ in the GS-formulas.

Exercise 3 (6 points): Orthogonal projections

(i) (2 points) Let $U \leq V$ be a subspace of a finite-dimensional inner product space V. Show that $P_{U^\perp} = I_V - P_U$.

(ii) (4 points) Let $U = \{x \in \mathbb{R}^3 \colon x + y + z = 0\}$. Find the matrix representation $A = \mathcal{M}(P_U)_{\mathcal{S},\mathcal{S}}$ of P_U with respect to the standard basis $\mathcal{S}$ of $\mathbb{R}^3$ and verify that $A^2 = A$.

Hint: Start by finding bases for U and $U^\perp$ and collect them in a basis $\mathcal{B}$ for $\mathbb{R}^3$. Now use the formula $\mathcal{M}(P_U)_{\mathcal{S},\mathcal{S}} = \mathcal{M}(I)_{\mathcal{B},\mathcal{S}}\mathcal{M}(P_U)_{\mathcal{B},\mathcal{B}}\mathcal{M}(I)_{\mathcal{S},\mathcal{B}}$. What is $\mathcal{M}(P_U)_{\mathcal{B},\mathcal{B}}$?