Relative entropies and their use in quantum information theory

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(based on my PhD thesis, arXiv:1611.08802)

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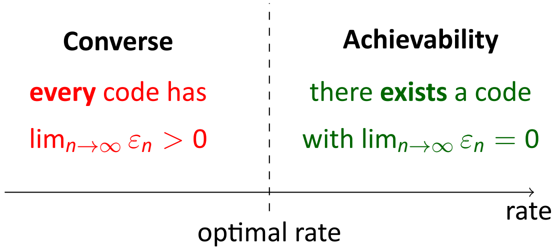
Motivation: Information theory basics

- Information theory is concerned with proving **coding theorems** of **information-processing tasks**.
- **Information-processing task:** Convert n uses/copies of some resource $\mathcal{R}$ into a "useful" resource of "size" x_n, which can be used to achieve the desired task up to an error ε_n.
- **Example:** Quantum source coding
  - $\mathcal{R}$: quantum state
  - x_n: # qubits of the compressed quantum state
  - Goal: asymptotically perfect retrieval of $\mathcal{R}$
- Depending on task, x_n should be minimal or maximal.

Motivation: Information theory basics

- If the error ε_n vanishes asymptotically in the number of uses ($n \to \infty$), we call $\lim_{n\to\infty} \frac{x_n}{n}$ an **achievable rate**.
- The **optimal rate** is the inf/sup over all achievable rates.
- **Coding theorem:** expresses the optimal rate as an **entropic quantity** (a function of the resource $\mathcal{R}$).



Motivation: von Neumann entropy

- In most known coding theorems in quantum information theory, the optimal rate is a function of the **von Neumann entropy**

$$S(A)_\rho := -\operatorname{Tr}(\rho_A \log \rho_A).$$

- Example: Quantum source coding [Schumacher 1995]

optimal compression rate $=$ von Neumann entropy

Motivation: von Neumann entropy

- Moreover, the von Neumann entropy serves as a building block for other information quantities such as:
  - **Conditional entropy** $S(A|B)_\rho := S(AB)_\rho - S(B)_\rho$

    $\rightarrow$ State merging, source coding with quantum side information
  - **Coherent information** $I(A\rangle B)_\rho := -S(A|B)_\rho$

    $\rightarrow$ Entanglement distillation, quantum information transmission
  - **Mutual information** $I(A;B)_\rho := S(A)_\rho + S(B)_\rho - S(AB)_\rho$

    $\rightarrow$ (Entanglement-assisted) classical information transmission
  - **Conditional Mutual information**

    $I(A;B|C)_\rho := S(AC)_\rho + S(BC)_\rho - S(C)_\rho - S(ABC)_\rho$

    $\rightarrow$ State redistribution

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First example: Quantum relative entropy

- The von Neumann entropy (along with its descendants) can be derived from a parent quantity, the **quantum relative entropy**:

$$D(\rho\|\sigma) := \mathrm{Tr}(\rho(\log\rho - \log\sigma)).$$

- We have:

$$S(A)_\rho = -D(\rho_A\|I_A)$$
$$S(A|B)_\rho = -D(\rho_{AB}\|I_A \otimes \rho_B)$$
$$I(A;B)_\rho = D(\rho_{AB}\|\rho_A \otimes \rho_B).$$

$I(A;B|C)_\rho$: not as straightforward

- **Question:** What does the quantum relative entropy tell us about quantum states?

First example: Quantum relative entropy

- $D(\cdot\|\cdot)$ is a **premetric** on the set of quantum states, that is, it is **positive definite**:

$$D(\rho\|\sigma) \geq 0, \text{ and } D(\rho\|\sigma) = 0 \text{ iff } \rho = \sigma.$$

- Hence, $D(\cdot\|\cdot)$ is a "distance" on the set of quantum states, but it is **not symmetric** and **does not satisfy triangle inequality**.

- Why do we still we care about this quantity then?

First example: Quantum relative entropy

- **Operational interpretation:** Quantum Stein's Lemma

  [Hiai and Petz 1991; Ogawa and Nagaoka 2000]

- Roughly: In asymmetric quantum hypothesis testing, $D(\rho\|\sigma)$ quantifies the minimal error of distinguishing between two given states ρ and σ (more later!).

- The larger $D(\rho\|\sigma)$ is, and thus the further ρ and σ are apart, the better we can distinguish between the two.

First example: Quantum relative entropy

- Moreover, for any quantum operation Λ evolving the system in the states ρ or σ, the following **data processing inequality** (DPI) holds:

$$D(\rho\|\sigma) \geq D(\Lambda(\rho)\|\Lambda(\sigma)).$$

- Meaning: two states **cannot become more distinguishable** after the system has evolved dynamically.

- DPI is the **key concept** for a relative entropy!

Relative entropies

Definition: Relative entropy

Any functional $\mathcal{D}(\cdot\|\cdot)$ on pairs of operators that satisfies:

- Data processing inequality: $\mathcal{D}(X\|Y) \geq \mathcal{D}(\Lambda(X)\|\Lambda(Y))$
- Positive definiteness on states.

- This is not an axiomatization, but rather a minimal set of requirements we have on a relative entropy.

- For free: unitary invariance $D(X\|Y) = D(UXU^\dagger\|UYU^\dagger)$

  $D(X\|Y) \geq D(UXU^\dagger\|UYU^\dagger) \geq D(U^\dagger UXU^\dagger U\|U^\dagger UYU^\dagger U) = D(X\|Y)$.

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Application: Strong subadditivity

- Crucial property of the von Neumann entropy:

$$S(AC)_\rho + S(BC)_\rho \geq S(ABC)_\rho + S(C)_\rho$$

[Lieb and Ruskai 1973]

- Equivalently: $I(A;B|C)_\rho \geq 0$.
- Very easy proof using data processing for $D(\cdot\|\cdot)$:

$$\begin{aligned} S(A)_\rho + S(BC)_\rho - S(ABC)_\rho &= D(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) \\ &\geq D(\rho_{AC}\|\rho_A \otimes \rho_C) \quad \text{(DPI with } \mathrm{Tr}_B\text{)} \\ &= S(A)_\rho + S(C)_\rho - S(AC)_\rho \end{aligned}$$

- Analysis of equality in this proof (i.e. states ρ_{ABC} with $I(A;B|C)_\rho = 0$) leads to the notion of **short Quantum Markov chains**. [Hayden et al. 2004]

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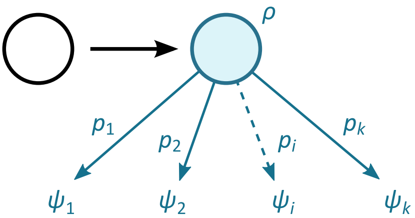
Application: Converse proofs

- Recall: A converse is one half of a coding theorem that gives a **fundamental limit** for asymptotically perfect protocols.
- Reformulation: For **all codes** beyond the optimal rate, the error has to be bounded away from zero.
- "Mathematical" part of the coding theorem.
- Use mathematical properties of relative entropies!

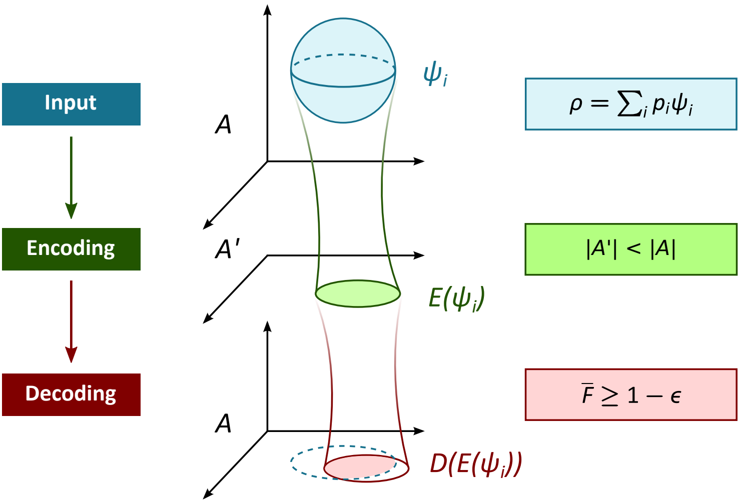
Revisiting: Quantum source coding



- **Quantum source**: emits pure states ψ_i with probability p_i.
- **Task**: compress signals by reducing dimension of supporting space (encoding $\mathcal{E}$).
- Original signals should be recoverable with some decoding operation ($\mathcal{D}$).
- **Figure of merit**: ensemble average fidelity

$$\bar{F} := \sum_i p_i \langle \psi_i | (\mathcal{D} \circ \mathcal{E})(\psi_i) | \psi_i \rangle.$$

Revisiting: Quantum source coding



Revisiting: Quantum source coding

- Simpler picture: n copies of quantum state $\rho_A = \sum_i p_i \psi_i$, purified by $|\varphi\rangle_{RA}$.
- Instead of ensemble average fidelity, consider trace distance of purification to final state of the protocol:

$$\varepsilon_n := \frac{1}{2}\|\varphi_{RA}^{\otimes n} - \mathcal{D}_n \circ \mathcal{E}_n(\varphi_{RA}^{\otimes n})\|_1.$$

- Stronger error criterion: $1 - \bar{F} \leq \varepsilon_n$.
- If $\varepsilon_n \to 0$, then $\lim_{n\to\infty} \frac{\log|A'|}{n}$ is an **achievable rate** for ρ_A.
- Schumacher's quantum source coding theorem:

$$\inf\{R\colon R \text{ is achievable for } \rho_A\} = S(A)_\rho.$$

Quantum source coding: Converse proof

- Schumacher's theorem (conv.): $\lim \frac{\log |A'|}{n} \geq S(A)_\rho$.
- Some more ingredients:
  - **Coherent information** as a relative entropy:

$$I(A\rangle B)_\rho = S(B)_\rho - S(AB)_\rho = D(\rho_{AB} \| I_A \otimes \rho_B).$$

  - Inherits a **data processing inequality**:

    For a quantum operation $\Lambda : B \to B'$,

$$\begin{aligned} I(A\rangle B)_\rho &= D(\rho_{AB} \| I_A \otimes \rho_B) \\ &\geq D(\mathrm{id}_A \otimes \Lambda(\rho_{AB}) \| I_A \otimes \Lambda(\rho_B)) \\ &= I(A\rangle B')_{\mathrm{id}_A \otimes \Lambda(\rho)}. \end{aligned}$$

  - Allows us to **undo** the (arbitrary) decoding and encoding!

Quantum source coding: Converse proof

- Some more ingredients:
  - Need to **relate** the **entropic quantity** (coherent information) to the **error of the protocol** (trace distance).
  - **Alicki-Fannes inequality:**
    If $\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon$, then
    $$|I(A\rangle B)_\rho - I(A\rangle B)_\sigma| \leq \log|A|\, f(\varepsilon),$$
    where $f(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$.

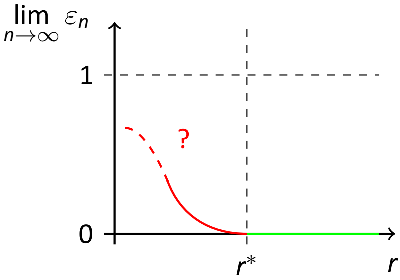
Quantum source coding: Converse proof

$$
\begin{aligned}
\log|A'| &\geq S(A')_{\mathcal{E}_n(\rho^{\otimes n})} \\
&\geq S(A')_{\mathcal{E}_n(\rho^{\otimes n})} - S(R^nA')_{\mathcal{E}_n(\varphi^{\otimes n})} \qquad (= I(R^n\rangle A')_{\mathcal{E}_n(\varphi^{\otimes n})}) \\
&= D(\mathcal{E}_n(\varphi_{RA}^{\otimes n}) \| I_R \otimes \mathcal{E}_n(\rho_A^{\otimes n})) \\
&\geq D(\mathcal{D}_n \circ \mathcal{E}_n(\varphi_{RA}^{\otimes n}) \| I_R \otimes \mathcal{D}_n \circ \mathcal{E}_n(\rho_A^{\otimes n})) \qquad \text{(DPI)} \\
&= I(R^n\rangle A^n)_{\mathcal{D}_n \circ \mathcal{E}_n(\varphi^{\otimes n})} \\
&\geq I(R^n\rangle A^n)_{\varphi^{\otimes n}} - n\log|R|\, f(\varepsilon_n) \qquad \text{(Alicki-Fannes ineq.)} \\
&\geq nI(R\rangle A)_\varphi - n\log|R|\, f(\varepsilon_n) \qquad \text{(additivity)} \\
&\geq nS(A)_\rho - n\log|R|\, f(\varepsilon_n)
\end{aligned}
$$

Hence: $\lim_{\varepsilon_n \to 0} \lim_{n\to\infty} \frac{\log|A'|}{n} \geq S(A)_\rho$.

Weak vs. strong converse

- "Problem" with converse so far: **no asymptotically perfect protocol** with a rate beyond the optimal one.
- In theory, this leaves room for trade-off between rate and error:

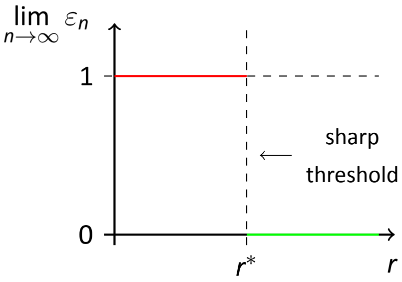


- Strong converse: No such trade-off possible!

Weak vs. strong converse

Strong converse

Every code with $r < r^*$ fails **with certainty**: $\varepsilon_n \xrightarrow{n\to\infty} 1$.



- Strongest form: $\varepsilon_n \geq 1 - \exp(-Cn)$ [Wolfowitz 1961]

Rényi entropies

- To prove a strong converse, we need something more flexible than the quantum relative entropy.
- One option: "Deform" $D(\cdot\|\cdot)$ in terms of a parameter $\alpha \geq 0, \alpha \neq 1$:

α-sandwiched Rényi divergence

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

[Müller-Lennert et al. 2013; Wilde et al. 2014]

Rényi entropies

α-sandwiched Rényi divergence

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

[Müller-Lennert et al. 2013; Wilde et al. 2014]

- Satisfies **DPI** for $\alpha \geq \frac{1}{2}$ and **positive definiteness**.
- **Limit property:** $\lim_{\alpha\to 1}\widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$.
  - Connects $\widetilde{D}_\alpha(\cdot\|\cdot)$ to $D(\cdot\|\cdot)$ and optimal rates.
  - "Ticket" to proving a strong converse!
- Usual Rényi entropy:

$$-\widetilde{D}_\alpha(\rho_A\|I_A) = S_\alpha(A)_\rho = \frac{1}{1-\alpha}\log\operatorname{Tr}\rho_A^\alpha$$

Quantum source coding: Strong converse proof

- To prove strong converse for quantum source coding, we need an equivalent of the Alicki-Fannes inequality.

- Define a Rényi version of the coherent information of ρ_{AB}:

$$\tilde{I}_\alpha(A\rangle B)_\rho := \min_{\sigma_B} \widetilde{D}_\alpha(\rho_{AB} \| I_A \otimes \sigma_B).$$

- For $\alpha \in [\frac{1}{2}, 1)$, take β such that $\frac{1}{\alpha} + \frac{1}{\beta} = 2$, then:

$$\frac{2\alpha}{1-\alpha} \log F(\rho_{AB}, \sigma_{AB}) \leq \tilde{I}_\beta(A\rangle B)_\rho - \tilde{I}_\alpha(A\rangle B)_\sigma.$$

[Leditzky et al. 2016]

Quantum source coding: Strong converse proof

- We can then prove for any $\alpha \in [\frac{1}{2}, 1)$ and all n:

$$F(\varphi_{RA}^{\otimes n}, \mathcal{D}_n \circ \mathcal{E}_n(\varphi_{RA}^{\otimes n})) \leq \exp\left[-n\frac{1-\alpha}{2\alpha}\left(S_\alpha(A)_\rho - \frac{\log|A'|}{n}\right)\right]$$

- Assume that we are in the "converse regime":

$$\lim_{n\to\infty} \frac{\log|A'|}{n} < S(A)_\rho$$

- Then there is an $\alpha_0 < 1$ such that for sufficiently large n,

$$S_{\alpha_0}(A)_\rho - \frac{\log|A'|}{n} \geq \delta > 0.$$

- Hence, $F(\varphi_{RA}^{\otimes n}, \mathcal{D}_n \circ \mathcal{E}_n(\varphi_{RA}^{\otimes n})) \leq \exp(-nC)$ with $C := \frac{1-\alpha_0}{2\alpha_0}\delta$.

Strong converse proof: Rényi entropy method

- Origin in classical IT, known as "Arimoto converse" for classical channel coding [Arimoto 1973]
- A similar method works well for proving strong converses for a number of QIT tasks:
  - Classical-quantum channels [Ogawa and Nagaoka 1999]
  - Classical information transmission through entanglement-breaking and Hadamard channels [Wilde et al. 2014]
  - Quantum information transmission through generalized dephasing channels [Tomamichel et al. 2017]
  - State redistribution [Leditzky et al. 2016]

Min-/Max-entropies and smooth entropies

- Rényi entropies serve as building block for the **min- and max-relative entropies**:

$$D_{\max}(\rho\|\sigma) = \inf\{\lambda\colon \rho \leq 2^{\lambda}\sigma\} = \lim_{\alpha\to\infty} \widetilde{D}_{\alpha}(\rho\|\sigma)$$
$$D_{\min}(\rho\|\sigma) = -2\log F(\rho,\sigma) = \widetilde{D}_{1/2}(\rho\|\sigma)$$

  [Renner 2005; Datta 2009; Tomamichel 2012]

- Considering their derived conditional entropies and smoothing over a ball of states ε-close to ρ_{AB}, we get the **smooth min- and max-entropies**.

- Key quantities in (classical and quantum) **cryptography** and **one-shot information theory**.

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Refining optimal rates

- Coding theorems determine the **optimal rate** of an information-processing task **in the asymptotic limit**.
- A strong converse gives a certain no-go theorem in the converse regime.
- But...
  - Asymptotic limit $n \to \infty$ mostly unrealistic $\longrightarrow$ we want good bounds for finite n!
  - For finite blocklength codes, how fast do we approach the optimal rate?
- Possible answer: Second order asymptotic expansions.

Quantum hypothesis testing

- Task: We are given an unknown quantum state, with the promise that it is either ρ (null hypothesis) or σ (alternative hypothesis).
- We can use any POVM (or **test**) $\{T, I - T\}$ to reach our conclusion (two-element POVMs suffice), where T corresponds to detecting ρ, and $I - T$ to σ.
- Two fundamental errors:
  - **Type-I error**: We infer σ instead of ρ (false accept).

$$\alpha(T) = \mathrm{Tr}((I - T)\rho)$$

  - **Type-II error**: We infer ρ instead of σ (false reject).

$$\beta(T) = \mathrm{Tr}(T\sigma)$$

Quantum hypothesis testing

- Since we cannot make both errors vanish at the same time, any optimal strategy will have a **trade-off** between α and β.

- **Asymmetric hypothesis testing**: Fix a threshold $\varepsilon \in [0,1]$ for the type-I error, and then minimize the type-II error:

$$\beta^* = \beta^*(\rho, \sigma, \varepsilon) = \min_{T \text{ test}} \{\beta(T) \colon \alpha(T) \leq \varepsilon\}.$$

- Define now the **hypothesis testing relative entropy**:

$$D_H^\varepsilon(\rho \| \sigma) := -\log \beta^* = -\log \min_{0 \leq T \leq I} \{\mathrm{Tr}(T\sigma) \colon \ \mathrm{Tr}(T\rho) \geq 1 - \varepsilon\}$$

- Satisfies **DPI** and **positive definiteness**!

Quantum hypothesis testing

- **Quantum Stein's Lemma**: Given i.i.d. states $\rho^{\otimes n}$ and $\sigma^{\otimes n}$, the optimal type-II error for any $\varepsilon \in [0, 1]$ is given by the quantum relative entropy:

$$\lim_{n\to\infty} -\frac{1}{n} \log \beta^*(\rho, \sigma, \varepsilon) = D(\rho\|\sigma).$$

- In terms of $D_H^\varepsilon(\cdot\|\cdot)$:

$$D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + o(n).$$

- Second order asymptotics: Determine the next order of n in the $o(n)$ term!
- **Second order expansion:**

$$D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n).$$

[Li 2014; Tomamichel and Hayashi 2013]

Quantum hypothesis testing

$$D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n).$$

- Second order corresponds to $\sqrt{n}$.
- **Quantum information variance**

$$V(\rho\|\sigma) := \mathrm{Tr}\left[\rho(\log\rho - \log\sigma)^2\right] - D(\rho\|\sigma)^2.$$

  Quantum version of variance of log-likelihood ratio.
- **Gaussian factor**

$$\Phi^{-1}(\varepsilon) := \sup\{z\colon \Phi(z) \leq \varepsilon\}.$$

  - Reason why second order expansions are often called **Gaussian approximation**.
  - Origin: Berry-Esseen Theorem, which gives speed of convergence in Central Limit Theorem.

Second order expansions of optimal rates

- How can we use second order expansion of a relative entropy for information-processing tasks?
- Strategy: Find finite-blocklength lower and upper bounds of the operational quantity of interest (e.g. $C(\rho_A) = \log |A'|$ for source coding) in terms of $D_H^\varepsilon(\cdot\|\cdot)$ and then expand.
- Schumacher's theorem (+ strong converse [Winter 1999]):

$$C(\rho_A) = nS(A)_\rho + o(n).$$

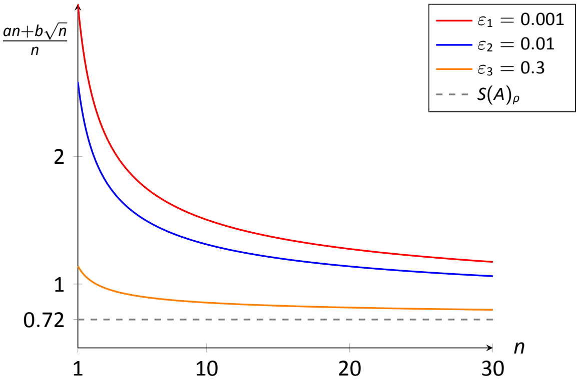
- Second order expansion (for *visible* quantum source coding):

$$C(\rho_A) = nS(A)_\rho - \sqrt{nV(\rho\|I)}\Phi^{-1}(\varepsilon) + O(\log n).$$

[Datta and Leditzky 2015]

Second order expansions for source coding

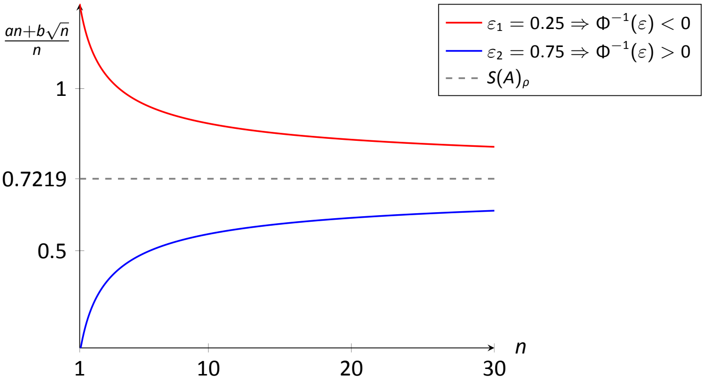
$$\log |A'| = nS(A)_\rho - \sqrt{nV(\rho\|I)}\Phi^{-1}(\varepsilon) + O(\log n).$$



$$\rho_A = \tfrac{1}{2}|0\rangle\langle 0|_A + \tfrac{1}{2}|+\rangle\langle +|_A \quad a = S(A)_\rho \quad b = -\sqrt{V(\rho_A\|I)}\Phi^{-1}.$$

Second order expansions of optimal rates

$$\log|A'| = nS(A)_\rho - \sqrt{nV(\rho\|I)}\Phi^{-1}(\varepsilon) + O(\sqrt{n}).$$



$$\rho_A = \tfrac{1}{2}|0\rangle\langle 0|_A + \tfrac{1}{2}|+\rangle\langle +|_A \quad a = S(A)_\rho \quad b = -\sqrt{V(\rho_A\|I)}\Phi^{-1}.$$

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Conclusion

- Relative entropies are functionals $\mathcal{D}(\cdot\|\cdot)$ on pairs of operators satisfying:
  - **Data processing inequality**: For any quantum operation Λ,
    $$\mathcal{D}(X\|Y) \geq \mathcal{D}(\Lambda(X)\|\Lambda(Y))$$
  - **Positive definiteness**: For quantum states ρ, σ,
    $$\mathcal{D}(\rho\|\sigma) \geq 0, \text{ and } \mathcal{D}(\rho\|\sigma) = 0 \text{ iff } \rho = \sigma.$$
- Their mathematical properties allow us to prove (strong) converse theorems and second order asymptotic expansions.

Conclusion

- **Quantum relative entropy**:

$$D(\rho\|\sigma) := \mathrm{Tr}(\rho(\log\rho - \log\sigma))$$

- **α-sandwiched Rényi divergence**, $\alpha \geq \frac{1}{2}, \alpha \neq 1$:

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1}\log\mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

satisfying $\lim_{\alpha\to 1}\widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$.

- **Hypothesis testing relative entropy**, $\varepsilon \in [0,1]$:

$$D_H^\varepsilon(\rho\|\sigma) = -\log\left(\min_{0\leq T\leq I}\{\mathrm{Tr}(T\sigma):\ \mathrm{Tr}(T\rho) \geq 1-\varepsilon\}\right)$$

satisfying $\lim_{n\to\infty}\frac{1}{n}D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma)$.

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Thank you very much for your attention!