Mapping Out the Quantum Channel Zoo

IGL Scholars: Ben Booker, Tianshun Gao, Anne Que, Yuxuan Wan, Lumi Xu

Graduate Mentor: Sujeet Bhalerao

Faculty Advisor: Prof. Felix Leditzky

Abstract: The goal of this project is to map out the "quantum channel zoo" by creating a website hosting a dynamic wiki-style database of the known quantum channels and their mathematical and information-theoretic properties. The idea is to create a valuable resource for scientists researching quantum channels.

1. Background

1.1 Density operators and partial trace

A **density operator** or **quantum state** is a linear operator $\rho \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ that is positive semidefinite and has trace 1.

Let $\{|i\rangle_B\}_{i=0}^{|B|-1}$ be an orthonormal basis of $\mathcal{H}_B$ (where $|B| = \dim \mathcal{H}_B$). The **partial trace** of an operator $X_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is defined as

$$\operatorname{tr}_B(X_{AB}) = \sum_{i=0}^{|B|-1} (\mathbb{1}_A \otimes \langle i|_B) X_{AB} (\mathbb{1}_A \otimes |i\rangle_B).$$

1.2 Quantum channels and unitary evolution in open systems

A **quantum channel** is a model for noise in an open quantum system S with environment E. It is defined mathematically as a **linear, completely positive, trace-preserving map** acting on density operators, and can be understood in physical terms as follows.

We assume that the environment E is initialized in a state $|0\rangle$, and the state $\rho_S \otimes |0\rangle\langle 0|_E$ on the whole system SE undergoes unitary evolution by $U_{SE}\colon \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_E) \longrightarrow \mathcal{L}(\mathcal{H}_{S'} \otimes \mathcal{H}_E)$:

$$\rho_S \otimes |0\rangle\langle 0|_E \mapsto U_{SE}(\rho_S \otimes |0\rangle\langle 0|_E) U_{SE}^\dagger.$$

Then the noisy quantum channel $\mathcal{N}\colon \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_{S'})$ is defined by considering the effective evolution on the system S (resp. S') alone, discarding the (inaccessible) environment:

$$\mathcal{N}(\rho_S) = \operatorname{tr}_E \left(U_{SE}(\rho_S \otimes |0\rangle\langle 0|_E) U_{SE}^\dagger \right).$$

2. Representations of quantum channels

We consider three representations of a channel $\mathcal{N}\colon A \to B$ with environment E:

- The **Choi operator** τ_{AB} of $\mathcal{N}\colon A \to B$ is defined as $\tau_{AB} = (\mathrm{id}_A \otimes \mathcal{N})(|\gamma\rangle\langle\gamma|_{AA'})$, where $|\gamma\rangle_{AA'} = \sum_{i=0}^{|A|-1} |i\rangle_A \bigotimes |i\rangle_{A'}$ is an unnormalized maximally entangled state on AA' defined in terms of an orthonormal basis $\{|i\rangle_A\}_{i=0}^{|A|-1}$ of $\mathcal{H}_A$. The channel action is given by
$$\mathcal{N}(\rho_A) = \operatorname{tr}_A \left[\tau_{AB} \left(\rho_A^T \otimes \mathbf{1}_B \right) \right].$$
The Choi operator of a quantum channel is a positive semidefinite operator satisfying $\tau_A = \operatorname{tr}_B \tau_{AB} = \mathbf{1}_A$
- A **Kraus representation** of a channel $\mathcal{N}$ consists of Kraus operators $\{K_i\}_{i=0}^{|E|-1}$ with $K_i\colon \mathcal{H}_A \to \mathcal{H}_B$ satisfying $\sum_{i=0}^{|E|-1} K_i^\dagger K_i = \mathbf{1}_A$ such that
$$\mathcal{N}(\rho_A) = \sum_{i=0}^{|E|-1} K_i \rho_A K_i^\dagger.$$
- A **channel isometry** $V\colon \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ satisfying $V^\dagger V = \mathbf{1}_A$ and
$$\mathcal{N}(\rho_A) = \operatorname{tr}_E \left(V \rho_A V^\dagger \right).$$
The isometric representation is closely related to the unitary evolution in the open system picture described in Sec. 1.2 above.

3. Examples of quantum channels

3.1 Depolarizing Channel

The depolarizing channel is a qubit channel defined as

$$\mathcal{D}_q(\rho) = (1-q)\rho + q \operatorname{tr}(\rho)\frac{\mathbf{1}}{2} = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z,$$

where $p \in [0,1]$ and $q = 4p/3$, and X, Y, Z are the Pauli matrices.

Its Kraus operators are

$$K_0 = \sqrt{1-p}\mathbf{1}, \quad K_1 = \sqrt{\frac{p}{3}}X, \quad K_2 = \sqrt{\frac{p}{3}}Y, \quad K_3 = \sqrt{\frac{p}{3}}Z.$$

Its channel isometry V acts as

$$|0\rangle_A \mapsto \sqrt{1-p}\,|0\rangle_B\,|0\rangle_E + \sqrt{\frac{p}{3}}|1\rangle_B\,|1\rangle_E + i\sqrt{\frac{p}{3}}|1\rangle_B\,|2\rangle_E + \sqrt{\frac{p}{3}}|0\rangle_B\,|3\rangle_E$$
$$|1\rangle_A \mapsto \sqrt{1-p}\,|1\rangle_B\,|0\rangle_E + \sqrt{\frac{p}{3}}|0\rangle_B\,|1\rangle_E - i\sqrt{\frac{p}{3}}|0\rangle_B\,|2\rangle_E - \sqrt{\frac{p}{3}}|1\rangle_B\,|3\rangle_E.$$

Its Choi state is

$$\tau_{AB} = \begin{pmatrix} 1-\frac{2}{3}p & 0 & 0 & 1-\frac{4}{3}p \\ 0 & \frac{2}{3}p & 0 & 0 \\ 0 & 0 & \frac{2}{3}p & 0 \\ 1-\frac{4}{3}p & 0 & 0 & 1-\frac{2}{3}p \end{pmatrix}.$$

3.2 Erasure Channel

The qubit erasure channel is defined for $\epsilon \in [0,1]$ by

$$\mathcal{E}_\epsilon(\rho) = (1-\epsilon)\rho + \epsilon\,|e\rangle\,\langle e|\,,$$

where $|e\rangle$ is an erasure flag state orthogonal to the input state ρ.

Its Kraus operators are

$$K_0 = \sqrt{1-\epsilon}(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|), \quad K_1 = \sqrt{\epsilon}\,|e\rangle\,\langle 0|\,, \quad K_2 = \sqrt{\epsilon}\,|0\rangle\,\langle 1|\,.$$

Its channel isometry V acts as

$$|0\rangle_A \mapsto \sqrt{1-\epsilon}|0\rangle_B|0\rangle_E + \sqrt{\epsilon}|e\rangle_B|1\rangle_E$$
$$|1\rangle_A \mapsto \sqrt{1-\epsilon}|1\rangle_B|0\rangle_E + \sqrt{\epsilon}|e\rangle_B|2\rangle_E.$$

Its Choi state is (0's are replaced by $\cdot$'s for readability)

$$\tau_{AB} = \begin{pmatrix} 1-\epsilon & \cdot & \cdot & \cdot & 1-\epsilon & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \epsilon & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1-\epsilon & \cdot & \cdot & \cdot & 1-\epsilon & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \epsilon \end{pmatrix}.$$

4. The Quantum Channel Zoo

4.1 Website

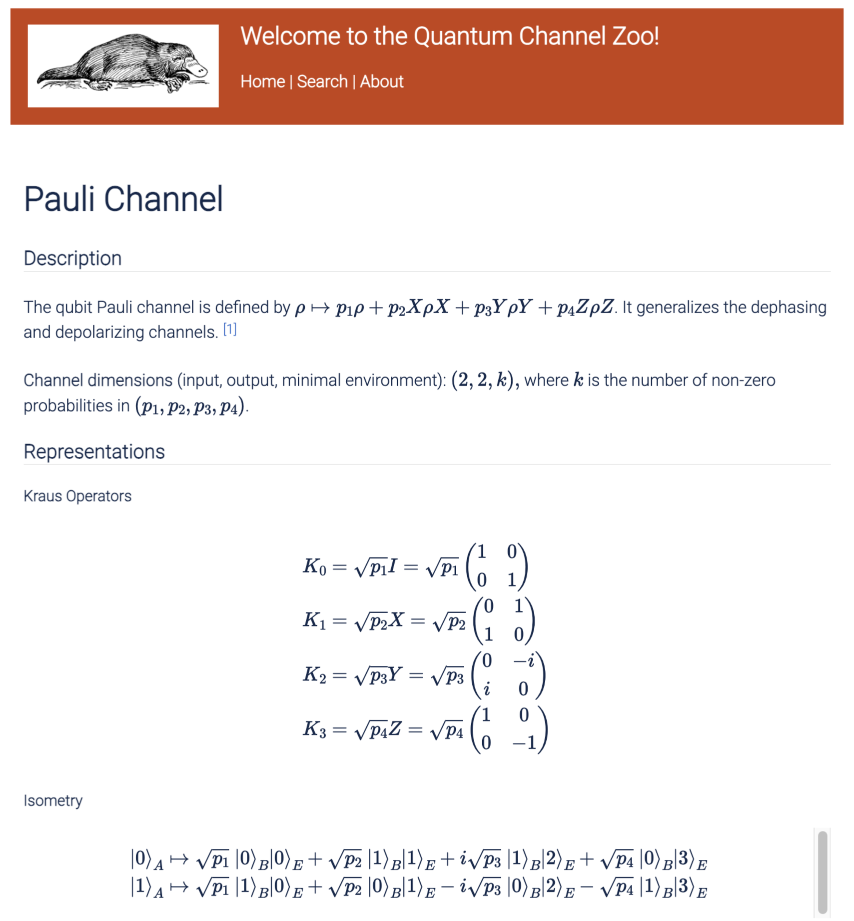
The goal of the website is to create a comprehensive list of quantum channels that clearly displays commonly cited channel properties, while also allowing for future additions and corrections. In this sense, the website mirrors a Wikipedia page. It is online at

`https://quantumchannelzoo.org/`

We have already implemented the following sections for a selection of quantum channels:

- Explicit form of the channel and a general description.
- The channel dimensions (input, output, minimal environment).
- Representations (Choi, Kraus, and Isometric).
- Parents/children of the given channel.
- References.

Currently, the database consists of the following channels: amplitude damping, generalized amplitude damping channel, multilevel and resonant multilevel amplitude damping channel, X-dephasing, Z-dephasing, and generalized dephasing channel, Pauli channel, dephrasure channel, qubit and higher dimensional erasure channel, qubit and higher dimensional depolarizing channel, and finally, the qutrit platypus channel along with its higher dimensional generalizations.



Scan this to visit quantumchannelzoo.org

4.2 Github repository

The website utilizes a (yet to be made public) GitHub repository where contributors will be able submit revisions and entries to the website. We use the ZooDB package created by Philippe Faist to build the website.

The Quantum Channel Zoo takes inspiration from the Error Correction Zoo [3] created by Philippe Faist and Victor V. Albert, which serves a similar purpose in documenting quantum and classical error correction codes in one centralized place. Both the Quantum Channel and Error Correction Zoo's fit into a larger trend within various scientific communities of creating these Zoos to consolidate information and make them easily accessible.

5. Future Work

- Finalize and beautify website, go public and invite collaborators
- Add information-theoretic and mathematical properties for channels (e.g., capacity information, entanglement properties, ...)
- Add more channels such as rocket channel, bosonic channels, ...

References

[1] Wilde, Mark M. Quantum Information Theory (2nd ed.). Cambridge: Cambridge University Press, 2017.
[2] Leditzky, F., Leung, D., Siddhu, V., Smith, G., and Smolin, J. A. (2023). The platypus of the quantum channel zoo. IEEE Transactions on Information Theory.
[3] The Error Correction Zoo. Edited by Victor V. Albert and Philippe Faist. URL: https://errorcorrectionzoo.org/

Acknowledgements: Support for this project was provided by the Illinois Geometry Lab, the Department of Mathematics at the University of Illinois at Urbana-Champaign.