Dephrasing channel and superadditivity of coherent information

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Outline

1  Quantum capacity of a quantum channel

2  (Anti)degradable channels and quantum capacity bounds

3  Dephrasure channel and its properties

4  Summary & Outlook
Entanglement generation

- Entanglement can be used as a resource in: teleportation, dense coding, entanglement-assisted communication, ...

- Assume Alice and Bob can communicate via a noisy quantum channel $\mathcal{N} : A \rightarrow B$.

- **Entanglement generation:** Use the noisy channel and local operations to generate entanglement between the parties.

- We can allow for **one-way classical communication** without changing the task.
Entanglement generation

Goal: Generate $m_n$ ebits $\ket{\Phi_+} \sim \ket{00} + \ket{11}$ through $n$ uses of the quantum channel $\mathcal{N}$.

Alice prepares $\ket{\psi}_{RA^n}$ and sends $A^n$ to Bob through $\mathcal{N}^\otimes n$.

Quantum capacity $Q(\mathcal{N}) := \sup \left\{ \lim_{n \to \infty} \frac{m_n}{n} \text{ s.t. } \varepsilon \to 0 \right\}$. 
Quantum capacity

▶ Coding theorem: [Lloyd 1997; Shor 2002; Devetak 2005]

\[ Q(N) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(N^\otimes n) \]  

where the channel coherent information \( Q^{(1)}(\cdot) \) is defined as

\[ Q^{(1)}(N) := \max_{\psi} I(A'B)(id \otimes N)(\psi). \]

with the coherent information \( I(P\.examples{\psi}Q)_{\rho} = S(Q)_{\rho} - S(PQ)_{\rho} \).

▶ Regularized formula \((*)\) in general intractable to compute.

▶ Notorious example: Qubit depolarizing channel

\[ D_p(\rho) := (1 - p)\rho + \frac{p}{3}(\rho X + Y \rho Y + Z \rho Z). \]

▶ Known: \( Q(D_0) = 1 \) and \( Q(D_p) = 0 \) for \( p \geq 0.25 \) (no-cloning).
Qubit depolarizing channel

- **Unknown:** $Q(D_p)$ for $p \in (0, 1/4)$.

- Partial answer for *low noise* ($p \gtrsim 0$):
  \[
  D_p \approx \text{id} \implies Q(D_p) \approx Q^{(1)}(D_p) \quad \text{up to } O(p^2 \log p)
  \]
  [FL, Leung, Smith 2017] based on [Sutter et al. 2017]

- **Superadditivity:** $Q^{(1)}(D_p) = 0$ for $p \geq 0.1894$, but
  $Q^{(1)}(D_p \otimes 3) > 0$ for $p \lesssim 0.1901$.
  [DiVincenzo et al. 1998]

- Achieved by repetition code $\sim |0\rangle \otimes^n + |1\rangle \otimes^n$ (degenerate code).

- Result: there are $\mathcal{N}$ and $n \in \mathbb{N}$ s.t. $Q^{(1)}(\mathcal{N} \otimes^n) > n Q^{(1)}(\mathcal{N})$.

- For which channels is superadditivity *not possible*, i.e.,
  $Q^{(1)}(\mathcal{N} \otimes^n) \leq n Q^{(1)}(\mathcal{N})$?
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Degradable and antidegradable channels

- **Complementary channel** $\mathcal{N}^c : A \rightarrow E$ associated to $\mathcal{N}$ models the leakage of information to the environment.

- Degradable channels have additive channel coherent information, $Q^{(1)}(\mathcal{N}^\otimes n) = nQ^{(1)}(\mathcal{N})$. [Devetak and Shor 2005]

- **Single-letter quantum capacity**: $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$. 

![Diagram of channels]
Degradable and antidegradable channels

- Antidegradable channels: $Q(\mathcal{N}) = 0$ due to no-cloning.
- Data-processing: $Q^{(1)}(\mathcal{N}) \leq 0$ for antidegradable channels.
- $\mathcal{D}_p$ is antidegradable for $p \geq 1/4$. 

<Diagram>

- Degradable: $\exists \mathcal{D}: B \rightarrow E$ s.t. $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$
- Antidegradable: $\exists \mathcal{A}: E \rightarrow B$ s.t. $\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$
Methods of bounding the quantum capacity

- Given a quantum channel that is not degradable/antidegradable, how can we bound its quantum capacity?
- If channel is almost degradable, then capacity $Q(\cdot)$ should be close to $Q^{(1)}(\cdot)$ → approximate degradability [Sutter et al. 2017]
- Give additional resources (NS/PPT-assistance) to the communicating parties that make quantities more "well-behaved". → SDP bounds [Leung and Matthews 2015; Wang et al. 2017]
- Decompose the channel into degradable/antidegradable parts and use their nice properties. → [Smith and Smolin 2008] [FL, Datta, Smith 2017]
Decomposition method

- Main insight: $Q(\cdot)$ is **convex** on channels with **additive** $Q^{(1)}(\cdot)$.
  
  [Wolf and Pérez-García 2007]

- This is true even if the channels in a decomposition are only completely positive, but not necessarily trace-preserving.

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**Upper bound on $Q(\cdot)$**

[FL, Datta, Smith 2018; Yang (in prep.)]

Let $\mathcal{N} = \sum_i p_i \mathcal{E}_i + \sum_i q_i \mathcal{F}_i$, where the $\mathcal{E}_i$ are **degradable** CP maps and the $\mathcal{F}_i$ are **antidegradable**. Then,

$$Q(\mathcal{N}) \leq \sum_i p_i Q^{(1)}(\mathcal{E}_i).$$

- This yields strongest upper bound on $Q(D_p)$ in high-noise regime (presented at BIID ’17).
Optimality of our bound

Main principle

For a channel $\mathcal{N} = (1 - \lambda)\mathcal{E} + \lambda\mathcal{F}$, with $\mathcal{E}$ degradable and $\mathcal{F}$ antidegradable, we only count degradable contributions:

$$Q(\mathcal{N}) \leq (1 - \lambda)Q^{(1)}(\mathcal{E}) = (1 - \lambda)Q^{(1)}(\mathcal{E}) + \lambda Q^{(1)}(\mathcal{F}) = 0$$

▶ Is there hope to improve our bound by also counting (negative) antidegradable contributions from a joint optimization?

$$(1 - \lambda)\max_\phi I(A\!\rangle\!B)_{\mathcal{E}(\phi)} + \lambda\max_\phi I(A\!\rangle\!B)_{\mathcal{F}(\phi)} \geq Q(\mathcal{N})$$

$\geq \sqrt{\text{convexity}} \max_\phi \left\{ (1 - \lambda)I(A\!\rangle\!B)_{\mathcal{E}(\phi)} + \lambda I(A\!\rangle\!B)_{\mathcal{F}(\phi)} \right\} \leq 0 \geq ?$$
Optimality of our bound

- Simple case: **flagged channel** \((\mathcal{E} \text{ deg.}, \mathcal{F} \text{ antideg.})\)
  \[
  \mathcal{N}_f = (1 - \lambda)\mathcal{E} \otimes |0\rangle\langle 0| + \lambda\mathcal{F} \otimes |1\rangle\langle 1| \tag{*}
  \]

- Bob can decide which channel occurred by first measuring flag.

- Easy to show:
  \[
  Q^{(1)}(\mathcal{N}_f) = \max_{\phi} \left\{ (1 - \lambda)I(A\rangle B)_{\mathcal{E}(\phi)} + \lambda I(A\rangle B)_{\mathcal{F}(\phi)} \right\}
  \]

- This is the conjectured upper bound on \(Q(\cdot)\)!

- Hence, **if true**, any channel \(\mathcal{N}_f\) of the form \((*)\) must have **additive coherent information**, since
  \[
  Q^{(1)}(\mathcal{N}_f) \leq Q(\mathcal{N}_f) \leq \text{conj. upper bound} = Q^{(1)}(\mathcal{N}_f).
  \]

- **Counterexample!** \(\longrightarrow\) Dephrasure channel
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Introducing: the dephrasure channel

**Dephrasure channel** \((\text{dephasing + erasure})\)

For \(p, q \in [0, 1]\),

\[
\mathcal{N}_{p, q}(\rho) := (1 - q)((1 - p)\rho + pZpZ) + q \text{Tr}(\rho)|e\rangle\langle e|.
\]

- Dephrasure channel is of the form
  
  \((1 - q)\ \text{deg.} + q \ \text{antideg.}\)

- **Flagged channel**, since \(\langle e|\rho|e\rangle = 0\) for all \(\rho\).

- Restrict to \(p, q \in [0, 1/2]\) from now on.

- \(\mathcal{N}_{p, q}\) is **simple but weird**: exhibits superadditivity of coherent information already for two uses of the channel.
Antidegradability of the dephrasure channel

- **Dephrasure channel:**
\[ \mathcal{N}_{p,q}(\rho) = (1 - q)((1 - p)\rho + pZ\rho Z) + q \text{Tr}(\rho)|e\rangle\langle e| \]

- **Complementary channel:**
\[ \mathcal{N}^c_{p,q}(\rho) = q\rho \oplus (1 - q) \sum_{x=0,1} \langle x|\rho|x\rangle|\phi_p^x\rangle\langle \phi_p^x|, \]

where \( |\phi_p^x\rangle = \sqrt{1 - p} |0\rangle + (-1)^x \sqrt{p} |1\rangle \).

- Complementary channel is also flagged!

- Want to construct antidegrading map \( \mathcal{A} \) such that
\[ \mathcal{N}_{p,q} = \mathcal{A} \circ \mathcal{N}^c_{p,q}. \]

- **Idea:** unambiguous state discrimination for \( \phi_p^x \).
Antidegradability of the dephrasure channel

Unambiguous state discrimination (USD):

- Input: two non-orthogonal states $|\psi_1\rangle$, $|\psi_2\rangle$ with $\langle \psi_1 | \psi_2 \rangle \neq 0$.

- Design POVM \{\Pi_1, \Pi_2, \Pi_?\} such that
  $$\langle \psi_1 | \Pi_2 | \psi_1 \rangle = \langle \psi_2 | \Pi_1 | \psi_2 \rangle = 0.$$  

- Hence, when receiving outcome ”1” or ”2” we are certain that we have $\psi_1$ or $\psi_2$.

- Have to abort if we get outcome ”?”.

- **Optimal measurement:** $\min \Pr(?) = | \langle \psi_1 | \psi_2 \rangle |$

[Ivanovic 1987; Dieks 1988; Peres 1988]
Antidegradability of the dephrasure channel

Strategy:

\[
\mathcal{N}_{p,q}^c(\rho) = q \rho \otimes |0\rangle\langle0|_F + (1 - q) \sum_{x=0,1} \langle x|\rho|x\rangle \varphi_p^x \langle \varphi_p^x | \otimes |1\rangle\langle1|_F
\]

\[\text{measure flag } F\]

\textbf{outcome } |0\rangle\langle0|: \]
\begin{align*}
\text{post-process } \rho \\
(\text{erase with prob. } 1 - \frac{(1-q)(1-2p)}{q})
\end{align*}

\textbf{outcome } |1\rangle\langle1|: \]
\begin{align*}
\text{USD to recover } \langle x|\rho|x\rangle \\
(\text{erase on ”?”})
\end{align*}

- Resulting map successfully degrades \(\mathcal{N}_{p,q}^c\) to
\[
\mathcal{N}_p,q(\rho) = (1 - q)((1 - p)\rho + pZ\rho Z) + q \text{ Tr}(\rho)|e\rangle\langle e|.
\]

- Map is completely positive iff \(q \geq (1 - q)(1 - 2p)\).
Antidegradability of the dephrasure channel

\[
q < \frac{1 - 2p}{2(1-p)}
\]

antidegradable
Single-letter coherent information

- For superadditivity of coherent information: need to know $Q^{(1)}(\mathcal{N}_{p,q})$.

- Erasure flag: easy to show that

$$Q^{(1)}(\mathcal{N}_{p,q}) = \max_{|\varphi\rangle_{AA'}} \{ (1 - q)I(A\rangle B)Z_p(\varphi) - qS(A)\varphi \},$$

where $Z_p(\rho)$ is the dephasing channel.

- Form of $Q^{(1)}(\mathcal{N}_{p,q})$ suggests that optimal state is diagonal in $Z$-basis $\rightarrow$ true! (simple calculus)

$$Q^{(1)}(\mathcal{N}_{p,q}) = \max_z \left\{ (1 - 2q)S\begin{pmatrix} (1+z)/2 & 0 \\ 0 & (1-z)/2 \end{pmatrix} \right\}$$

$$- (1 - q)S\begin{pmatrix} 1-p & z\sqrt{p(1-p)} \\ z\sqrt{p(1-p)} & p \end{pmatrix} \right\}$$
Single-letter coherent information

\[ Q^{(1)}(\mathcal{N}_{p,q}) = \max_{|\phi\rangle_{AA'}} \left\{ (1 - q)I(A\rangle B)z_p(\varphi) - qS(\varphi_A) \right\}. \]
Superadditivity of coherent information

- First thing to try... **weighted repetition code:**

\[
|\varphi_n\rangle = \sqrt{\lambda} |0\rangle_R |0\rangle_A^n + \sqrt{1-\lambda} |1\rangle_R |1\rangle_A^n
\]

- For \( n = 1 \), this is the optimal single-letter code.

- \( \mathcal{N}_{p,q}^\otimes n = ((1-q)\mathcal{Z}_p + q \text{ Tr}(\cdot) |e\rangle\langle e|)^\otimes n \):

  sum of channels of the form \( \mathcal{Z}_p^\otimes k \otimes (\text{ Tr}(\cdot) |e\rangle\langle e|)^\otimes n-k \).

- Coherent information splits up into different erasure patterns, since

  \[
  S(\sum_i p_i \rho_i \otimes |i\rangle\langle i|) = \sum_i p_i S(\rho_i) + H(\{p_i\}).
  \]

- Repetition code: all partial erasures cancel.

- Easy: compute action of dephasing \( \mathcal{Z}_p^\otimes n \) on repetition code.
Superadditivity of coherent information

- (Almost) closed formula for repetition code ($\varphi_n = \varphi_n(\lambda)$):

\[
Q^{(1)}(\varphi_n, \mathcal{N}_{p,q}^\otimes) = \left( (1 - q)^n - q^n \right) h(\lambda) - (1 - q)^n \left( 1 - u \operatorname{artanh} u - \frac{1}{2} \log \left( 1 - u^2 \right) \right).
\]

- $u = u(\lambda, p, n) = \sqrt{1 - 4\lambda(1 - \lambda)(1 - (1 - 2p)^{2n})}$.

- $h(\lambda) = -\lambda \log \lambda - (1 - \lambda) \log(1 - \lambda)$ is the binary entropy of $\lambda$.

- To get superadditivity, maximize over parameter $\lambda$ (weight of logical $|0\rangle$ in the repetition code).
Superadditivity of coherent information

- Weighted repetition code $|\varphi_n\rangle := \sqrt{\lambda}|0\rangle \otimes^{n+1} + \sqrt{1-\lambda}|1\rangle \otimes^{n+1}$.
- Plot for $n = 2, 3$ of the non-negative part of
  \[ \frac{1}{n} \max_{\lambda} I(A^n B^n)_{\mathcal{N}_{\rho, q}(\varphi_n)} - Q^{(1)}(\mathcal{N}_{\rho, q}) \]
Superadditivity of coherent information

- Weighted repetition code $|\varphi_n\rangle := \sqrt{\lambda}|0\rangle^{\otimes n+1} + \sqrt{1-\lambda}|1\rangle^{\otimes n+1}$.
- Plot for $n = 4, 5$ of the non-negative part of

$$\frac{1}{n} \max_{\lambda} I(A^{\otimes n}B^n)_{N_{pq},q(\varphi_n)} - Q^{(1)}(\mathcal{N}_{pq})$$

![Graphs showing the non-negative part of the function for $n = 4, 5$.](image)
Superadditivity of coherent information

- Superadditivity also holds in the "extreme form":

\[ Q^{(1)}(N_{p,q}) = 0 \quad \text{but} \quad Q^{(1)}(\varphi_n, N_{p,q}^\otimes n) > 0 \]

for \( n \geq 2 \) and some \( p, q \rightarrow \text{increased threshold} \).

- We also have more elaborate codes achieving superadditivity, for example for \( n = 3 \):

\[
|\chi_3\rangle := |00\rangle|00\rangle \otimes |\psi_1\rangle + |11\rangle|11\rangle \otimes |\psi_1\rangle
+ |01\rangle|01\rangle \otimes |\psi_2\rangle + |10\rangle|10\rangle \otimes X|\psi_2\rangle,
\]

for some pure states \( |\psi_i\rangle \).

- We also found good codes using a neural network state ansatz.

[\text{Bausch, FL; arXiv:1806.08781}]

- Not clear whether optimal codes for \( n \geq 2 \) are diagonal in \( Z \)-basis (true for \( n = 1 \)).
Superadditivity of coherent information

- Plot for $\mathcal{N}_{p,3p}$ along diagonal $(p, 3p)$
- Threshold is increased by repetition codes $\varphi_n$. 

![Graph showing the superadditivity of coherent information with thresholds for different repetition codes $\varphi_n$.](image-url)
Numerical optimization techniques

▶ Easy observation: \( I(A \rangle B) \psi \otimes \mathcal{N}(\phi) = 0 \) for a product input state \( |\psi\rangle_A \otimes |\phi\rangle_{A'} \).

▶ Hence, many local maxima in high-noise regime where most states have negative channel coherent information.

▶ Gradient is likely to get stuck \(\longrightarrow\) gradient-free optimization?

▶ Many biology-inspired examples: genetic algorithms, artificial bee colonization, particle swarm optimization (PSO)

▶ Idea of PSO:
  ▶ Send out \( N \) particles, each probing the landscape.
  ▶ Each particle records personal best function value, and all know the global swarm best.
  ▶ In each iteration, particle velocity is updated with weights towards personal best, global best, and inertial movement.
Particle swarm optimization

Ackley function: 

\[ f(x, y) = -20 \exp \left( -0.2 \sqrt{0.5(x^2 + y^2)} \right) - \exp \left[ 0.5 \left( \cos 2\pi x + \cos 2\pi y \right) \right] + e + 20 \]
Particle swarm optimization

Ackley function: \( f(x, y) = -20 \exp \left[ -0.2 \sqrt{0.5(x^2 + y^2)} \right] - \exp \left[ 0.5(\cos 2\pi x + \cos 2\pi y) \right] + e + 20 \)
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Particle swarm optimization

Ackley function: $f(x, y) =\]
$$\begin{align*}
&-20 \exp \left[ -0.2 \sqrt{0.5 (x^2 + y^2)} \right] - \exp \left[ 0.5 (\cos 2\pi x + \cos 2\pi y) \right] + e + 20
\end{align*}$$

Iteration: 21
Particle swarm optimization

Ackley function: $f(x, y) =$

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Iteration: 27
Particle swarm optimization

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Private capacity of a quantum channel

- **Private capacity** $P(\mathcal{N})$: highest rate of private classical communication between Alice and Bob

- Coding theorem: \cite{Devetak2005, Cai2004}

\[
P(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} P^{(1)}(\mathcal{N}^{\otimes n})
\]

with the **private information**

\[
P^{(1)}(\mathcal{N}) = \max_{\{p_i, \rho_i\}} [I(X; B)_{\rho} - I(X; E)_{\rho}].
\]

- Private information can be superadditive, \cite{Smith2008}

\[
P^{(1)}(\mathcal{N}^{\otimes n}) > nP^{(1)}(\mathcal{N}).
\]
Private capacity of a quantum channel

- Quantum information transmission is necessarily private:
  \[ P(\mathcal{N}) \geq Q(\mathcal{N}). \]

- Also holds for information quantities: \( P^{(1)}(\mathcal{N}) \geq Q^{(1)}(\mathcal{N}) \).

- For degradable channels: [Smith 2008]
  \[ P(\mathcal{N}) = Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) = P^{(1)}(\mathcal{N}). \]

- For antidegradable channels: \( P(\mathcal{N}) = 0 = Q(\mathcal{N}) \).

- There are channels with \( Q(\mathcal{N}) = 0 \) and \( P(\mathcal{N}) > 0 \) (e.g. entanglement-binding channels).

- Leads to superactivation of \( Q(\cdot) \): \( \exists \mathcal{N}_1, \mathcal{N}_2 \) with \( Q(\mathcal{N}_i) = 0 \) but \( Q(\mathcal{N}_1 \otimes \mathcal{N}_2) > 0 \). [Smith and Yard 2008]
Numerical investigations suggest the following is an optimal private ensemble:

\[ \rho_1 = \frac{1}{2}, \quad \rho_1 = \lambda |+\rangle \langle +| + (1 - \lambda) |-\rangle \langle -| \]

\[ \rho_2 = \frac{1}{2}, \quad \rho_2 = \lambda |-\rangle \langle -| + (1 - \lambda) |+\rangle \langle +| \]

\[ |\pm\rangle \sim |0\rangle \pm |1\rangle \]
Separation of private and coherent information

- Separation of capacities? $P(N_{p,q}) > Q(N_{p,q}) \ (= 0)$?

- Superadditivity of private information?
  $P^{(1)}(N_{p,q}^\otimes n) > nP^{(1)}(N_{p,q})$?
Summary

- Discussed an upper bound on quantum capacity of a channel based on decomposition into degradable and antidegradable channels.

- Question of optimality leads to the dephrasure channel

$$\mathcal{N}_{p,q}(\rho) = (1 - q)((1 - p)\rho + pZ\rho Z) + q \text{Tr}(\rho)|e\rangle\langle e|.$$ 

- Dephrasure channel checks a lot of marks on "weirdness chart":
  - superadditivity of coherent information for two uses
  - separation of private and coherent information
  - possible superadditivity of private information?
Outlook

- Some open questions:
  - Formula for single-letter private information (needed to show superadditivity)?
  - Tight upper bounds on quantum capacity?
  - Increase threshold up to region of antidegradability?

- Further effects of superadditivity? → superadditivity in classical communication with limited entanglement assistance (ongoing work with Elton Zhu, Quntao Zhuang)

- Can we understand the superadditivity of coherent information in the light of the recent work on $\alpha$-bit capacities?
References


Yang, D. Manuscript in preparation.

Thank you very much for your attention!
QIP 2019: Jan 14-18, 2019 at University of Colorado Boulder

Website: jila.colorado.edu/qip2019
Local organizers: Felix Leditzky, Graeme Smith
Program committee chair: Matthias Christandl
Submission deadline: sometime in September (TBD)